Static Equilibrium Analysis of Mechanical Systems with Compliant Members

by

UJWAL VINAYAK JOSHI

DEPARTMENT OF MECHANICAL ENGINEERING
INDIAN INSTITUTE OF TECHNOLOGY KANPUR

February, 2000

Static Equilibrium Analysis of Mechanical Systems with Compliant Members

A thesis Submitted

in Partial Fulfilment of the Requirements

for the Degree of

Master of Technology

by

UJWAL VINAYAK JOSHI

to the

DEPARTMENT OF MECHANICAL ENGINEERING

INDIAN INSTITUTE OF TECHNOLOGY KANPUR

February, 2000

CERTIFICATE

It is certified that the work contained in the thesis entitled **Static Equilibrium Analysis of Mechanical Systems with Compliant Members** by **Ujwal Vinayak Joshi**, has been carried out under my supervision and this work has not been submitted elsewhere for a degree.

Bhaskar Dasgupta

Dr. Bhaskar Dasgupta,

Assistant Professor,

Department of Mechanical Engineering,

Indian Institute of Technology,

Kanpur. 208016.

February, 2000.

Acknowledgement

I express my deep sense of gratitude to my thesis supervisor Dr. Bhaskar Dasgupta for his excellent guidance, invaluable suggestions, constant support and encouragement during the tenure of this thesis work. It was a great pleasure to work under him as a lot of care with personal touch was rendered.

I wish to acknowledge all the friends and faculty members, who were a part of the process of intellectual enrichment and personal growth in many aspects of life.

Ujwal Vinayak Joshi

Dedicated

to

my parents

Abstract

Mechanical systems (mechanisms and structures) find numerous applications in mechanical engineering. They have a number of applications in the field of automobiles, aviation, manufacturing tools, instrumentation etc. Hence, the Computer Aided Analysis (kinematic, static and dynamic) of such systems plays a vital role in designing these systems for a particular application.

Much of the analysis is done considering the members of the system rigid. But in real life systems, the compliance of many of the members plays an important role and hence, cannot be neglected.

In this thesis, we have attempted the static equilibrium analysis of compliant mechanical systems, under the action of external loading. These are the statically determinate systems in which the mobility of the system is due the compliant members only, i.e. if the compliant members are considered as rigid, the entire mechanical system will behave as a structure. The basic methodology used is to develop the governing constraint equations for rigid and deformable members and then using them in the force equilibrium equation to get the deformed configuration of the system at equilibrium. This method is tested on some planar and spatial mechanical systems.

Our approach requires a less number of generalized coordinates of the system for the analysis and hence is computationally fast, efficient and accurate. Also it is a generalized approach, applicable to any planar as well as spatial mechanical system.

Contents

List of Figures

Chapter 1

Introduction

A mechanical system is defined as a collection of interconnected rigid bodies or links that can move relative to each other, consistent with joints that limit relative motion of pairs of bodies. Mechanical systems can be broadly classified into two types: mechanisms, with degrees of freedom (DOF) greater than zero and structures, with DOF zero (or less).

These mechanical systems find a tremendous number of applications in the mechanical field, viz. in I.C. engines, suspension systems of automobiles, governors, automobile transmissions, shapers, robots, structural supports etc. Thus, the breadth of applications being extensive, the analysis and design of mechanical systems has an important place in mechanical engineering. This analysis helps the designers to decide upon the final geometry and other parameters, to make the system suitable for particular type of application.

1.1 Computer Aided Analysis of Mechanical Systems

The analysis (kinematic, static or dynamic) of mechanical systems, involves a large number of non-linear governing equations of the system. Thus, the analytical solutions to the analysis problem are applicable only to a few relatively simple systems. Because of this, the mechanical designer traditionally resorted to graphical techniques and physical models for analysis. But these methods are limited in generality and rely on the designer's intuition.

The mathematical models of static and dynamic systems are many a times solved by "clever formulation" that takes advantage of the properties of a specific system to obtain simplified forms of constraint equations. The ingenious selection of position and orientation variables can occasionally lead to a formulation with independent variables that allow manual derivation, but rarely any analytical solution of the equations of the system. Thus, the "clever formulation" approach is also limited to relatively simple mechanical systems.

The solution to the highly non-linear equations for any general mechanical system requires the help of advanced numerical and optimization techniques. These numerical techniques, though able to give the correct analysis, require a tremendous computation, which is manually impossible . But with the advent of high-speed computers, using the Computer Aided Analysis, this tedious task is now left to the computer, saving a considerable time of the designer. It also helps him in experimenting with the parameters of the system, to choose

the best suitable combination, for the particular application. Thus, today, Computer Aided Analysis plays a very important role in the design of the mechanical systems.

The main objective of the Computer Aided Analysis of mechanical systems is to create a formulation and computer software that allow the engineer to

1. input data, that define the mechanical systems of interest and automatically formulate governing equations for analysis

2. automatically solve non-linear equations of the system, and

3. provide computer graphics output of results of simulations to communicate results to the designer or analyst.

One category of the mechanical systems is that of systems consisting of rigid links and compliant members, in which the mobility of the system is due to the compliant members only, i.e. if the compliant members are also considered as rigid, the entire mechanical system will behave as a structure. A few examples of such systems are as shown in figure 1.1. In all these examples, the compliant members of the system are represented by springs.

The compliant members can be: various types of springs, like the linear, torsional or leaf springs, bushings, elastic membranes, bellows, gaskets etc. The various examples of such systems are automobile suspension systems, spring controlled governors, spring loaded instruments, earth movers, supporting structures etc. Even, the pressure cooker with flexible gasket and a bicycle with inflated tyres, are some examples in a broad sense.

1.2 Static Equilibrium Problem

When a set of external forces and moments are applied to the mechanical system, the compliant members deform and the system changes its position and orientation, i.e. its configuration, and attains a final equilibrium configuration. The questions that arise are (1) Does the system attains a final configuration? (2) If so, what is the equilibrium configuration ?

The solution to the above questions is basically the static equilibrium analysis of the mechanical system under consideration. The static equilibrium of a mechanical system can be well visualized and understood with the help of the figure 1.2.

The external loading on the system includes, the externally applied loads like forces due to gas pressure and the body forces i.e., the weights of the various links or members of the systems and other forces which include friction and damping forces that act due to the relative motion between the components of the system.

This static equilibrium analysis will be helpful in the design of members of such mechanical systems, keeping in mind the deformation caused in the deformable members due to the action of the actual external forces. The designer will get the freedom of experimenting with the system parameters like link lengths, stiffnesses, materials etc. of the members in order to choose the best possible solution.

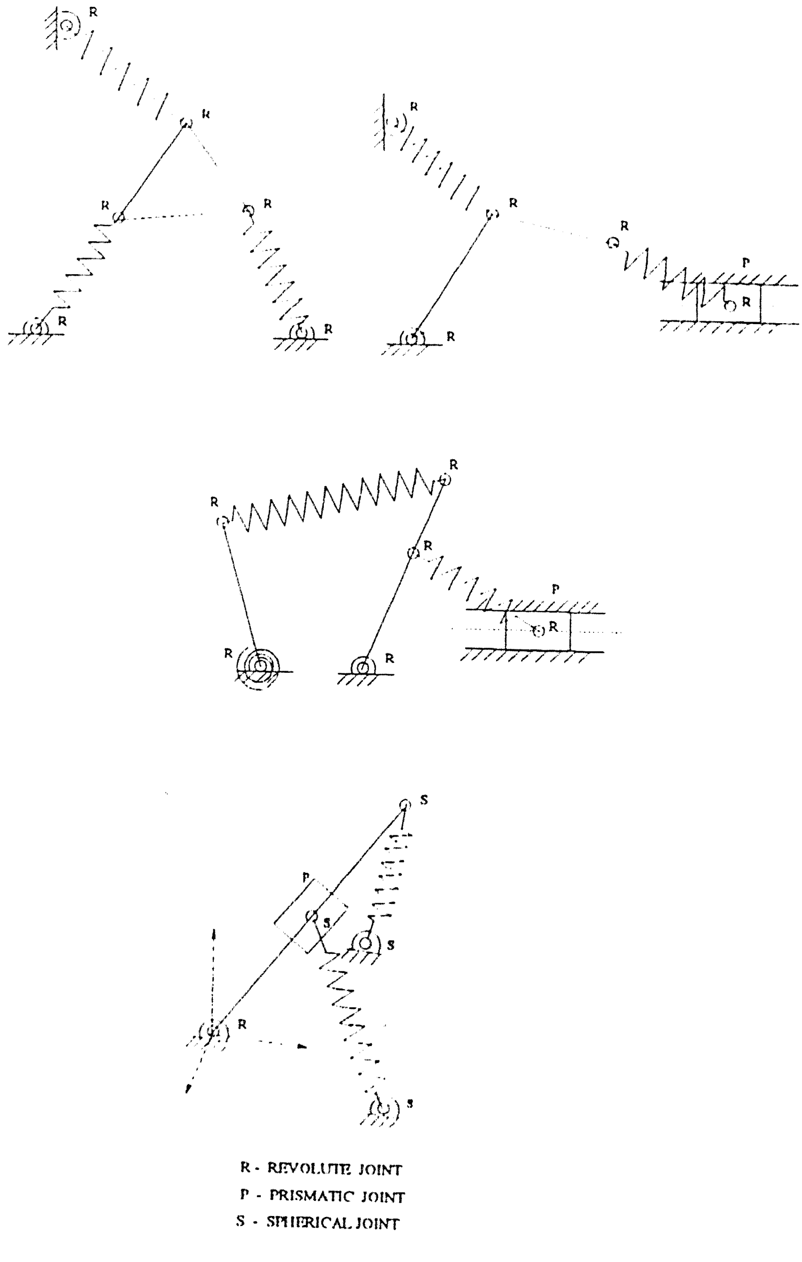


Figure 1.1: mechanical systems with compliant members

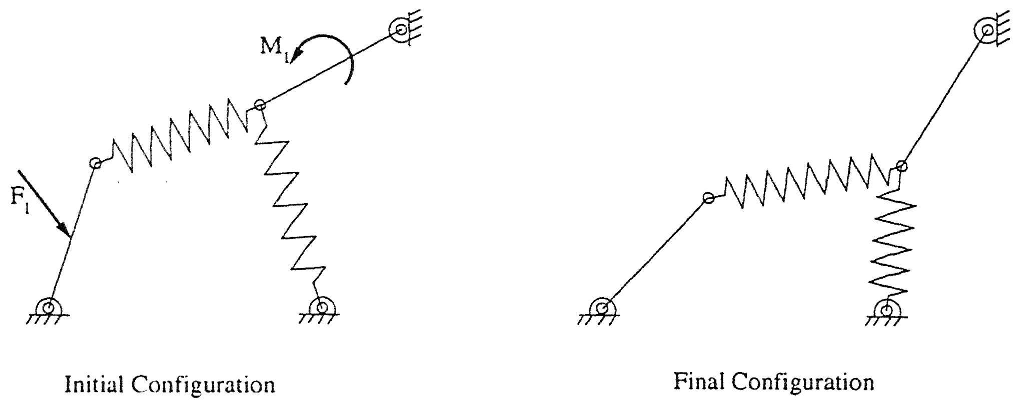


Figure 1.2: Static equilibrium of compliant mechanical system

1.3 Literature Review

Here, various previous works carried out in the field of compliant mechanical systems are discussed.

Salamon [1] introduced a methodology for compliant mechanism design that used a pseudo-rigid-body model of compliant mechanisms with the compliance modeled as torsional and linear springs. The purpose of the pseudo-rigid-body model is to provide a means by which a compliant mechanical system can be modeled as a rigid-body system. This allows the well known rigid-body analysis to be used for the analysis of compliant systems. These models are much easier to analyze than idealized models which required finite element or elliptic integral solutions. Pseudo-rigid-body model generally have closed form displacement and force equations that allow the designer to more easily see the effects of parameters on system design. Thus, it simplifies the design process.

Howell and Midha [2] presented a paper in which they have used loop-closure theory for the analysis and synthesis of compliant mechanisms. They also modeled the system with pseudo-rigid-body model. The work represents the first known effort at presenting a loop-closure methodology for the analysis and synthesis of compliant mechanical systems.

A paper presented by Midha, Howell and Norton [3] reviews the pseudo-rigid body model of a compliant mechanism. Its use in the determination of limit positions of a compliant mechanism is also investigated for the first time.

Hac [4] derived finite element model for flexible planar linkages. Large mechanism motion is studied by using truss-type elements and is a combination of rigid body displacement and elastic longitudinal deformation of links. The method is illustrated for the case of planar linkages modeled with modified (two dimensional) truss elements and with beam finite elements.

Lowen and Chassapis [5] have presented a review of work done on the elastic behavior of linkages. The reviewed publications have been grouped according to their basic premises: analytical methods, finite element methods, optimization schemes and general experimentation.

1.4 Scope of the Thesis

This thesis deals with the static equilibrium of planar and spatial mechanical systems with compliant members. These mechanical systems are user-defined, i.e. the user will specify or input the system in terms of its parameters like number of links, link-lengths, types of joints, connectivity of links etc. and the external loading applied on it.

The thesis is divided into two parts. In the first part, on the basis of approximate information about the system, provided by the user, the initial correct configuration of the system is obtained. In the second part, the equilibrium analysis of the system under the given loading, is carried out, to get its final equilibrium configuration.

1.5 Assumptions

In this approach, to solve the equilibrium problem, following assumptions are made to reduce the complexity of the problem.

1. The mechanical system under consideration is statically determinate. Static determinacy is explained in the formulation of the problem.

2. Compliant members are considered as linear elastic elements, i.e. there is a linear relationship between the forces applied and the deformations of these members and this behavior is independent of the magnitude of deformation.

3. Only binary flexible links are considered. Rigid links may have any number of connections.

4. Friction due to the relative motion of links, at joints, is neglected. Also the damping present at joints and in the links is neglected.

5. Joint clearances are neglected.

6. All the links, which under the actual loading, will not have any significant deformation, are considered rigid.

7. The loading applied on the system is time independent and stationary i.e. the position of the forces remain the same during the deformation.

1.6 Organization of the Thesis

In the next chapter, complete formulation of the static equilibrium analysis is explained. Chapter three deals with the loop closure equations. These equations are used in our approach, for developing the governing kinematic constraint equations. Chapter four covers the assembly part. The purpose of this assembly is to correct the approximate system parameters provided by the user. Optimization technique is used to assemble the mechanical system. In this chapter, development of the constraint equations with the help of loop closure

equations is explained. Also the calculation of Jacobian and Hessian of these equations is covered in this chapter.

Chapter five covers the overall static equilibrium analysis procedure. In this chapter, development of deformable constraint equations and its derivatives is discussed. Finally, the application of the force equilibrium method for the analysis of the mechanical system is explained.

In chapter six, some representative results of assembly and static equilibrium analysis of planar and spatial systems are reported to establish the capability of the method. Conclusion of the thesis is presented in chapter seven with possible future scope of work.

Chapter 2

Formulation

In this chapter, a general formulation of the static equilibrium problem is explained which is developed on the methodology of Dasgupta et al [6].

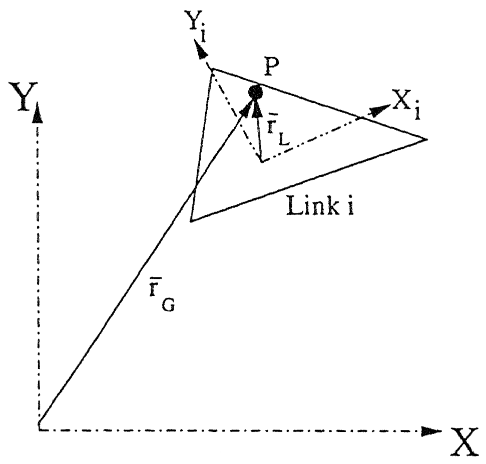
2.1 Generalized Coordinates

For a given mechanical system, first a set of generalized coordinates $X \subset R^n$, where n is the number of generalized coordinates, is identified to describe the system completely. Any set of variables that uniquely specifies the position and orientation of all bodies in the system, i.e., the configuration of the system, can be taken as the set of generalized coordinates (g.c.). Generalized coordinates may be independent (i.e. free to vary arbitrarily) or dependent (i.e. required to satisfy some constraint equations). For the systems in motion (mechanisms), some of the g.c. are independent, varying only with time. But in our case, the system being a structure, the g.c. are dependent.

2.2 Reference Systems

There are two types of reference frames for a mechanical system. The global reference frame and the local or body-fixed reference frame. In the global reference frame, the coordinates of all the links of the mechanical system are expressed with respect to the common reference point of global origin. In the local reference frame, the parameters of every link are expressed with respect to the local origin of the link, which may be any point on the link. These frames are as shown in figure 2.1.

Here, the ground connection of the first link of the system, is considered as the origin of the global reference system and centre of gravity (c.g.) of every link, as the local origin of the corresponding link, as shown in the figure 2.2.



X,Y - Global reference frame
X, Y - Local reference frame
r_G - Global location of point P
r_L - Local position of point P w.r.t. link i

Figure 2.1: Global and local reference frames

2.3 Selection of Generalized Coordinates

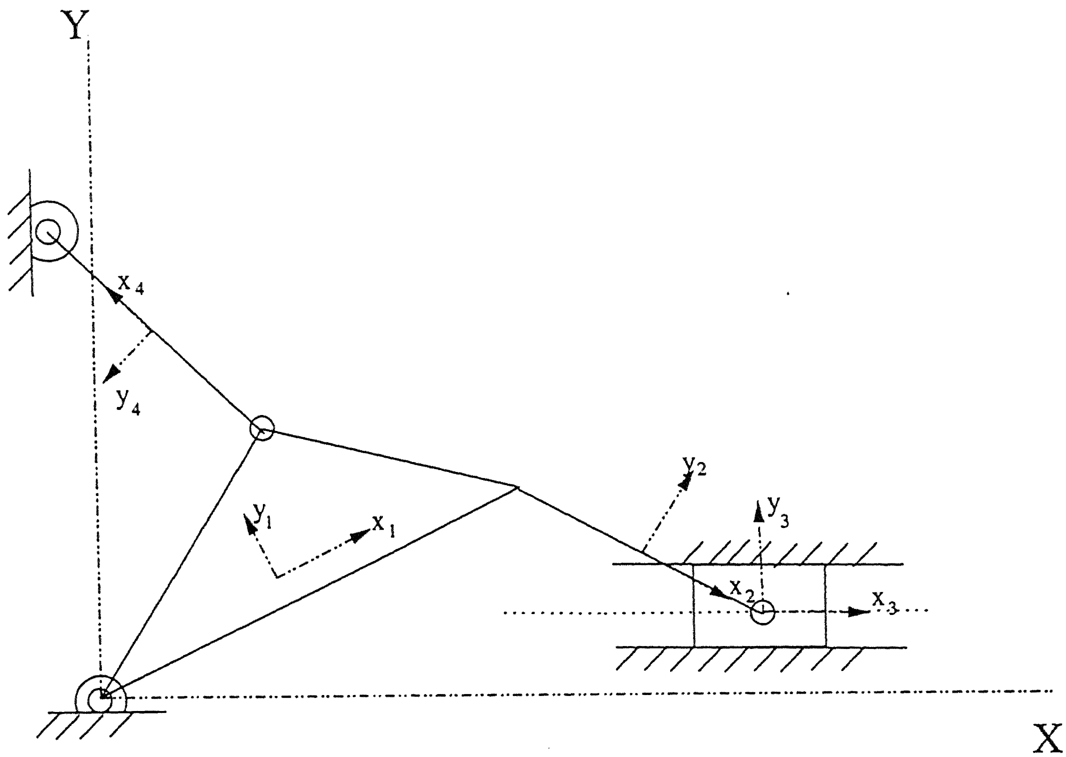
The selection of the g.c. for the system, plays as important role in the generality of the approach and the computational need. The different standard sets of g.c. that can be used for getting the governing equations required for the formulation are

1. the global x, y, z coordinates of local reference frame and its orientation w.r.t the global reference system, i.e. Cartesian g.c. of each link,
2. the absolute angles of each link with respect to the global reference system, and
3. the relative orientations of the link with respect to the previous link.

All the three types of g.c. for a small planar system are shown in the figure 2.3.

The first set is used for planar as well as spatial mechanical systems. But the number of g.c. per link are large (3 for planar, 6 for spatial), which makes it computationally inefficient and also affects the computational accuracy, as the solution requires additional numerical methods like center difference method to get the derivatives of the set of governing equations, which are required during the analysis. The second set handles a lesser number of g.c., but with a drawback that it is well suited for planar systems only and faces problems in complex spatial systems.

The third set of relative orientations combines the advantages of the first two sets. It requires a less number of g.c. for the system. The equilibrium analysis requires the differentiation of the governing constraint equations. This set of g.c. gives the derivatives of the



X , Y - Global reference frame

x_i , y_i - Local reference frames

Figure 2.2: Global and local reference frames

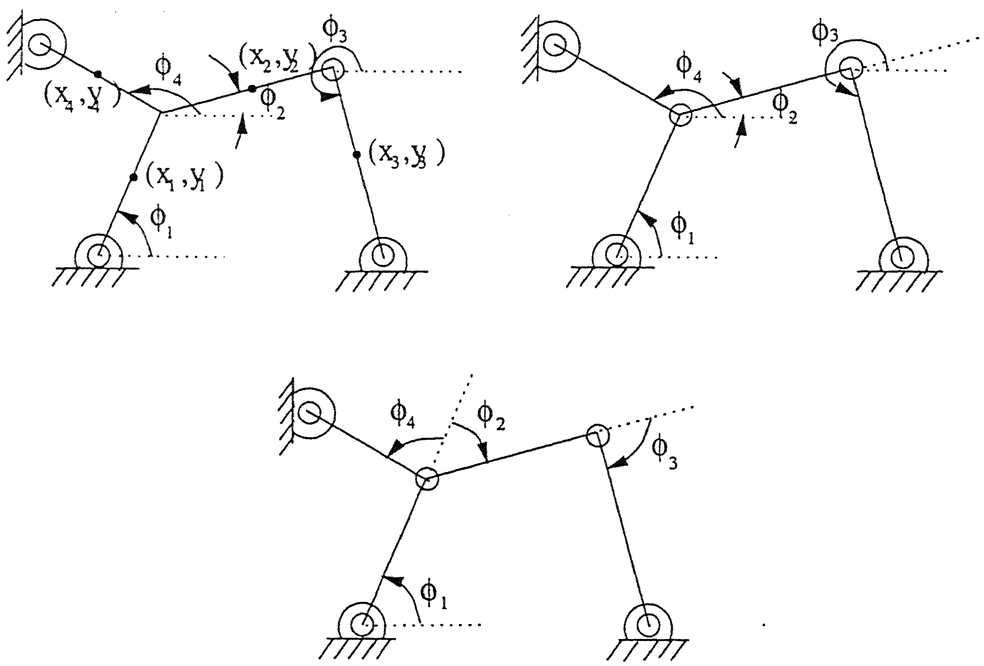


Figure 2.3: Various sets of generalized coordinates

governing equations, in a simplified way, as explained in the next chapter, where the loop closure equations will be explained, which act as the backbone of our approach. Thus, the selection of third set of g.c. makes the analysis computationally fast, efficient and accurate.

In the next step, the constraints (if any) among the g.c. are developed in the form

$$f(X) = 0 \quad f \in R^m; \tag{2.1}$$

where m = number of rigid constraints

i.e.

$$f_1(\phi_1, \phi_2, \phi_3, ...\phi_n) = 0$$

$$f_2(\phi_1, \phi_2, \phi_3, ...\phi_n) = 0$$

$$- - - - - - - -$$

$$f_m(\phi_1, \phi_2, \phi_3, ...\phi_n) = 0$$

where $X = [\phi_1, \phi_2, \phi_3, ...\phi_n]^T$ is the set of generalized coordinates.

These constraint equations are due to the various joints in the system, that restrict the motion of the links or relative motion between a pair of links. These constraint equations must imply the geometry of the joint. This is essential in the computer simulation. That is, the equations must be satisfied yielding the geometrical relations due to the joints. Otherwise, the mathematical model fails to define the configuration of the system, during and after the deformation.

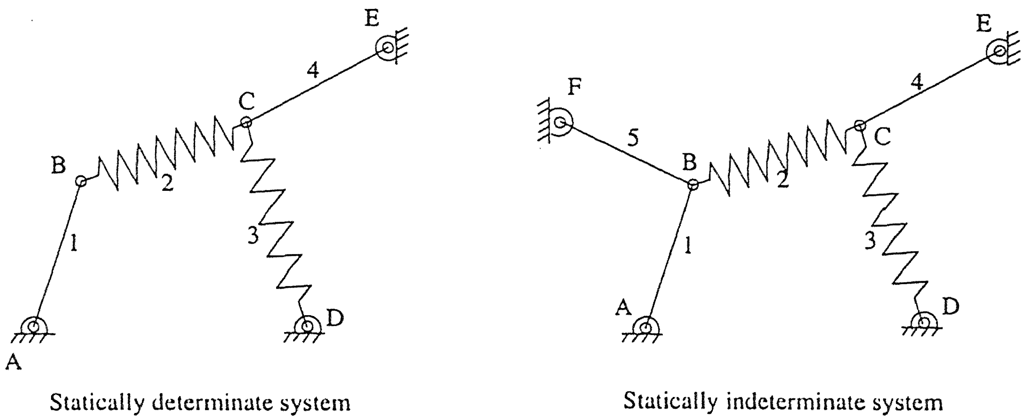


Figure 2.4: Statically determinate and indeterminate cases

2.4 Deformable Coordinates

The deformable members of the actual mechanical system are modeled as linear or torsional springs or a combination of the two.

The deformable coordinates for these members of the actual mechanical systems can be their lengths (or angles for torsional springs) or the deformations under the action of loading. We will be using the lengths of the linear springs and angles of the torsional springs in the system, as the deformable coordinates.

These deformable coordinates $\theta \epsilon R^p$, p = number of deformable coordinates, are then related to the g.c. as

$$\theta = g\,(X) \tag{2.2}$$

where $g \epsilon R^p$.

2.5 Static Determinacy

As the mechanical system under consideration is constrained and its mobility is owing to the deformable members only, we have

$$m + p \geq n$$

The cases for which the equality holds (m+p=n) are statically determinate cases and those for which inequality holds (m+p>n) are the statically indeterminate cases. Statically indeterminate systems have redundant constraints. The examples of both the cases are shown in figure 2.4.

Every link in a planar system has 3 DOF. In example 1, there are 2 rigid links (links 1 and 4), hence we have n = 6. There are 2 deformable members, link 2 and link 3, making the deformable coordinates, p = 2. Every revolute joint connecting the rigid links, gives two

constraints. Here, the number of revolute joints, connecting only the rigid links, are two, viz. A and E. Hence, the number of rigid members, $m = 2\times 2 = 4$.

Therefore, in this case, we have,

$$m + p = n$$

Hence it is a statically determinate system.

In the second example, number of rigid links are three, viz. links 1, 4 and 5. Hence, the number of g.c. is $n = 9$. Number of deformable links (p) is 2 (links 2 and 3). The number of revolute joints connecting only rigid links are 4 (joints A, B, E and F) making m=8.

Thus, in this case, we have

$$m + p > n$$

making the structure statically indeterminate.

This method is used, in the thesis, to determine whether the system given by the user, is statically determinate or indeterminate.

The statically indeterminate cases do have physical relevance and can be solved, but in this thesis, we are concerned only with statically determinate systems.

2.6 Generalized Force

The external loading is constituted by the actual stationary forces and moments on the system at various locations with respect to the global reference frame. Also, the weights of all the links of the system constitute a part of the loading. For the static equilibrium, these forces are considered to be at the fixed locations and along the fixed orientations, with respect to the global reference frame, throughout the deformation of the system. Also this loading is independent of time. In other words, the analysis is quasi-static.

The applied load, F_{actual} is transformed to the generalized force $F_{load} \in R^n$ corresponding to the g.c. This transformation is explained in detail, in the chapter 5, in which the static equilibrium procedure is explained.

These externally applied forces cause internal reaction forces, in the members of the system. The forces at the rigid constraints (given by equation 1), are denoted by $\lambda \in R^m$, while the forces at the deformable members are denoted by $F_\theta \in R^p$.

This F_θ is related to the deformations through constitutive relations as

$$F_\theta = K\,\Delta\theta \tag{2.3}$$

where $K \in R^{p\times p}$ is the stiffness matrix and $\Delta\theta$ is the set of deformations in every deformable member, under the action of external loading. For discrete elements, the stiffness matrix K is often diagonal (decoupled). In addition, for linear elastic constitutive relations (as considered in the thesis), it will be constant. Thus, we have

$$F_\theta = \begin{bmatrix} K_1 & 0 & - & 0 \\ 0 & K_2 & - & 0 \\ - & - & - & - \\ 0 & 0 & - & K_p \end{bmatrix} \begin{Bmatrix} \theta_1 \\ \theta_2 \\ - \\ \theta_p \end{Bmatrix}$$

2.7 Static Equilibrium Equation

There are many approaches by which this static equilibrium problem can be tackled. Those are, minimum total energy method, virtual work or variational method and force equilibrium method.

The first two methods involve complicated non-linear equations which result in complex computational techniques for solving them. But these methods can analyze both statically determinate and indeterminate systems. The third approach, i.e. the force equilibrium method, also involves non-linear equations. But, the nature of these equations is simple, and hence they are easy to solve, which makes the analysis compuationally fast and efficient. This approach, though, is suitable only for the statically determinate problems. Since we are dealing with the statically determinate systems, we have opted for the force equilibrium approach.

According to the force equilibrium approach, at equilibrium, the algebraic sum of the external forces applied on the system and internal forces generated in the system, is zero, i.e.

$$F_{external} + F_{internal} = 0$$

Thus, for the mechanical system under consideration, we have

$$F_{load} + \left\{ \left(\frac{\partial f}{\partial X}\right)^T \lambda + \left(\frac{\partial g}{\partial X}\right)^T F_\theta \right\} = 0; \tag{2.4}$$

Equations 2.1-2.4 are the equations among the unknowns $X, \theta, F_\theta, \lambda$, to be solved with the known load F_{actual}, to determine the equilibrium configuration. In particular, the solution for X is required to describe the final equilibrium configuration.

2.8 Singularity

Consider the 5-bar parallelogram model, as shown in figure 2.5. It is a statically determinate mechanical system. But, if a force F is applied on the system, there is no force in the opposite direction to balance it and hence the system will continuously keep on moving. Thus, the system cannot achieve a stable equilibrium configuration. Such behavior of the system is called the singularity. The question that arises during the equilibrium analysis, is how to detect this singularity?

Here, if we define

$$\phi(X) = \begin{bmatrix} f(X) \\ g(X) - \theta \end{bmatrix}, \qquad \phi \,\epsilon\, R^n \tag{2.5}$$

and analyze the rank of the Jacobian

$$\frac{\partial \phi}{\partial X} = \begin{bmatrix} \frac{\partial f}{\partial X} \\ \frac{\partial g}{\partial X} \end{bmatrix},$$

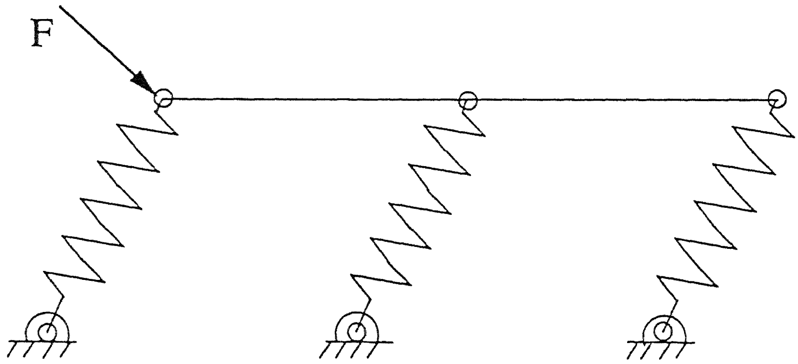


Figure 2.5: Singular configuration

we can make a priori estimate regarding the behavior of the system. As long as the Jacobian is full-rank, we can expect the structure to attain an equilibrium configuration. If the Jacobian is found to be rank-deficient, then the structure possesses unconstrained mobility and its behavior will be unpredictable to some extent.

2.9 Solution Scheme

The system of equations 2.1-2.4 consists of a total number of $m + n + 2p$ scalar equations, in as many unknowns. In principle, the entire system can be solved together, but that entails a large amount of computation. Utilizing the particular structure of the equations, we can decompose the problem into smaller modules by the algorithm below.

Step 1: $X = X_{intitial}$, $\theta = \theta_{initial}$
Step 2: From equation 4, solve for λ and F_θ.
Step 3: From this F_θ , use equation (3), to determine θ
If θ converged, then **STOP**.
Step 4: For this θ, solve $\phi(X) = 0$ for X.
Go to Step 2.

In the above procedure, step 2 is essentially the solution of a linear system. The nature of step 3 depends upon the characteristics of the deformable members. In our case, this again is a set of linear equations. Step 4 involves the solution of a system of nonlinear equations.

This is the overall formulation and algorithm used for the static equilibrium analysis of any arbitrary mechanical system, to get its final configuration.

Chapter 3

Loop Closure Equations

In this approach, the governing equations are developed by using the loop closure equations. Hence it is necessary to understand the theory of these loop closure equations. The loop closure equations are covered in details in Sandor et al [7].

The mechanical system under consideration, may have a number of kinematic loops as shown in figure 3.1.

Every loop starts from the ground connection of the link at which the global reference frame is fixed and ends at the next ground connection. In example 1, loop 1 starts from base of link 1 and ends at the base of link 3 (ground connection B). Now, the position of the ground connection B with respect to global origin is known. This creates two constraints equations for x and y coordinates of the system along the loop.

In case of a prismatic joint at the end of the loop, the two constraints are the position of slider along a fixed direction and the orientation of the slider.

Thus, in planar systems, two constraint equations are developed per ground connection with respect to the global origin.

In case of spatial mechanical systems, three or more constraints are present per ground connection, depending upon the type of the joint with which the system is connected to the ground. This is explained later in the chapter four (section 4.3).

3.1 Link and Joint Modeling with Elementary Matrices

The construction of constraint equations, using the loop closure equations, needs the elementary matrices for modeling links and joints of the structure. Since, we have to deal with planar as well as spatial mechanical systems, we use the homogeneous transformation matrices to get the geometric relation given by every joint of the system.

Before modeling the joints, the local reference or coordinate systems on the two links must be defined. In the figure 3.2, these coordinate systems are denoted as (x_i,y_i,z_i) and (x_j,y_j,z_j). The x-axis of every local coordinate system, is along the direction of connection between the two links and the links revolve about the z-axis as shown.

Link i and revolute joint ij are described using the homogeneous rotation matrix about z-axis. Also, prior to the rotation about this joint ij, there is a translation of $L = [l_x, l_y, l_z, 1]$

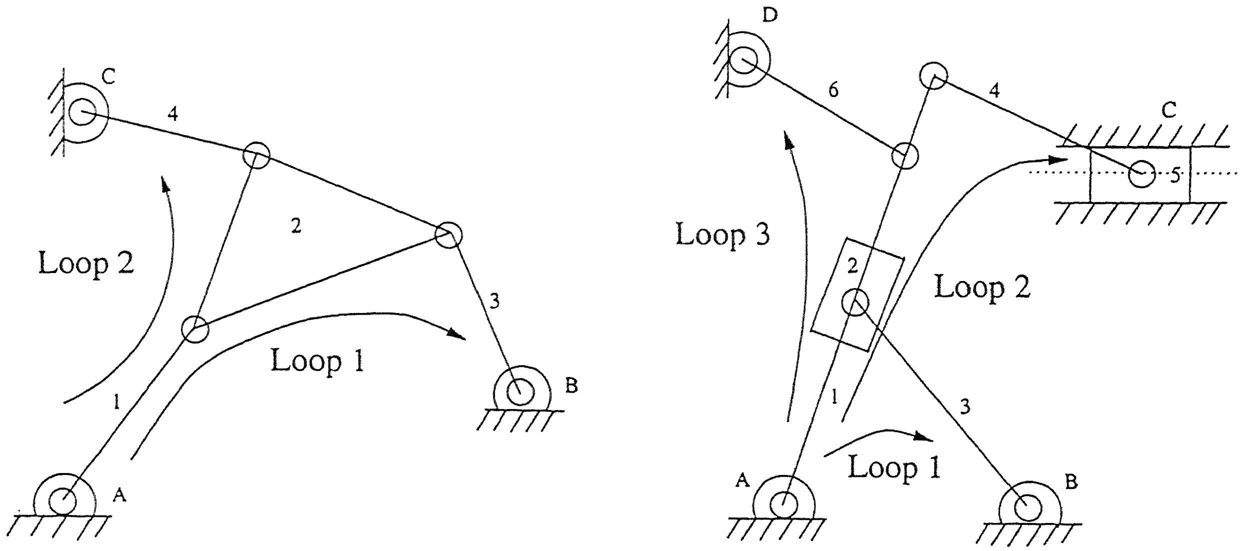


Figure 3.1: Kinematic loops in a mechanical system

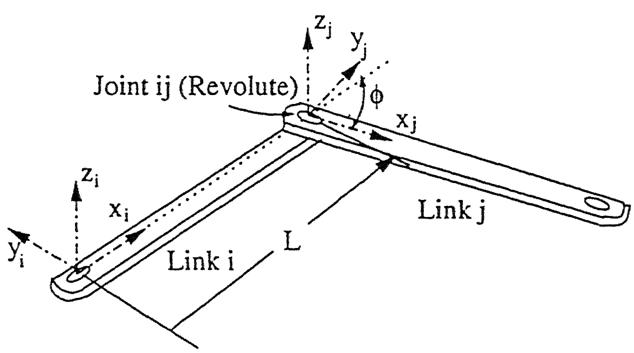


Figure 3.2: Revolute joint

from local reference frame of link i to that of j. Therefore, the relation of of the j-th link with respect to the i-th link is

$$T_1 = T(L) * R(\phi)$$

$$= \begin{bmatrix} 1 & 0 & 0 & l_x \\ 0 & 1 & 0 & l_y \\ 0 & 0 & 1 & l_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} cos\phi & -sin\phi & 0 & 0 \\ sin\phi & cos\phi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} cos\phi & -sin\phi & 0 & l_x \\ sin\phi & cos\phi & 0 & l_y \\ 0 & 0 & 1 & l_z \\ 0 & 0 & 0 & 1 \end{bmatrix} = T(L, \phi)$$

In this case, $l_y = l_z = 0$.

Thus, $T(L, \phi)$ is the homogeneous transformation, describing the revolute joint. The first quantity in the parentheses is the translational distance and second is the amount of rotation with respect to previous link.

With this transformation matrix, we can now express the coordinates on the j-th link with respect to i-th link.

In a similar way, the other joints can be modeled as simple concatenated homogeneous transformations. This representation for other types of joints is given in the appendix.

3.2 Modeling Mechanical Systems Using Loop Closure Equations

The modeling of systems using loop closure equations, to get the governing kinematic constraint equations, consists of simple matrix multiplication of the homogeneous transformations, along each loop in the system.

A four-bar mechanism is a single loop mechanical system. Though it is not a structure, it is used as an example to explain the loop closure equation easily. It is as shown in the figure 3.3.

Here, (x_0, y_0, z_0) is the global coordinate system. Links 1, 2 and 3 have local coordinate systems (x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3) respectively, fixed to them. The system has all revolute joints. Consequently, the (x_1, y_1, z_1) coordinate system, is defined relative to the global coordinate system by

$$T_1 = T(0, \phi_1)$$

where ϕ_1 is defined in the system 0. Furthermore, the (x_2, y_2, z_2) coordinate system is defined relative to the (x_1, y_1, z_1) system by

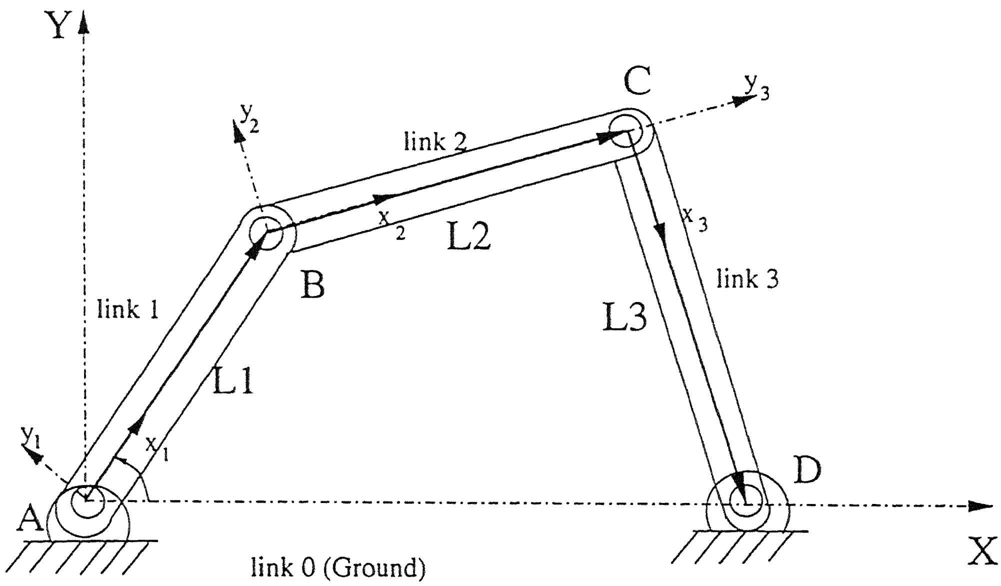


Figure 3.3: Four-bar (single loop) mechanism

$$T_2 = T(L_1, \phi_2)$$

where L_1 and ϕ_2 are defined in the system 1. In a similar fashion,

$$T_3 = T(L_2, \phi_3)$$

where L_2 and ϕ_3 are defined in system 2.

However, this does not close the loop for the mechanism. But, as the ground connection is reached, this matrix multiplication is sufficient, as far as getting the constraints is considered. The accurate location of the end ground connection in this loop is already known as $D = [x, y, z, 1]^T$.

Thus, we have the constraint equations as

$$f = T(0, \phi_1) * T(L_1,\phi_2) * T(L_2, \phi_3) * \{L_3\} - \{D\} = \{0\} \tag{3.1}$$

This gives two constraint equations for the planar mechanical systems (x and y). Also ϕ_1, ϕ_2, ϕ_3 being the generalized coordinates, these constraints are expressed as functions of these generalized coordinates, using this loop-closure method.

3.3 Taking Derivatives of the Loop Closure Equations

The loop closure equations are used for constructing the constraint equations. For the equilibrium analysis, we need the governing constraint equations. For the equilibrium analysis,

we need the Jacobian and Hessian of the these constraint equations (as will be discussed later). This necessitates differentiation of the loop closure equations.

As seen in the equation (1), any one g.c. of the system described by the loop closure equation, is wholly contained in only one of the matrices making up the product. Thus, while taking the derivatives of loop closure equations with respect to a g.c., say ϕ_1, only the components in the matrix having ϕ_1, are changed to the derivatives with respect to ϕ_1, and the rest of the matrix product remains unchanged.

Thus, we have,

$$\frac{\partial f}{\partial \phi_1} = \frac{\partial}{\partial \phi_1}[T(0,\phi_1)] * T(L_{1,}\phi_2) * T(L_2,\phi_3) * \{L_3\}$$

$$= T'(0,\phi_1) * T(L_{1,}\phi_2) * T(L_2,\phi_3) * \{L_3\}$$

Similarly, the derivatives with respect to all the g.c. can be calculated.

In the same way, the second derivatives with respect to any g.c., say ϕ_1, can be calculated as

$$\frac{\partial^2 f}{\partial \phi_1^2} = \frac{\partial^2}{\partial \phi_1^2}[T(0,\phi_1)] * T(L_{1,}\phi_2) * T(L_2,\phi_3) * \{L_3\}$$

$$= T''(0,\phi_1) * T(L_{1,}\phi_2) * T(L_2,\phi_3) * \{L_3\}$$

In this way, with the help of loop closure equations, any planar or spatial system can be easily modelled in terms of its g.c. The calculation of derivatives of the constraint equations with respect to g.c. also becomes simple using the loop closure equations.

Chapter 4

Assembly

The procedure for the static equilibrium analysis is divided into two parts:

1. Assembly
2. Equilibrium

The assembly phase gives the user, freedom to input the approximate relative orientations of all the members of the mechanical system. The correction of user input for the mechanisms, using the assembly can be found in Haug [8]. In simple planar cases, it is easier for the user to give accurate data of the orientation, but in the complicated planar and spatial structures, this task is quite difficult. Hence, in this phase, the system is assembled by obtaining the correct relative orientations, satisfying all the constraints with the use of optimization technique.

If the initial estimates provided by the user are grossly incorrect, and the system fails to assemble, the user is informed that an infeasible design has been specified. If the system has a singular configuration, it is reported to the user, stopping the further analysis of the system.

In the equilibrium phase, static equilibrium analysis of the assembled configuration is performed and the final equilibrium configuration is displayed as a graphical output. Here also, the singularity checks are provided in order to tackle with the situations, in which the system attains a singular configuration during its deformation.

The entire procedure is explained using the following 4-link (apart from the fixed frame) planar structure, with two deformable members, link 2 and link 4, as shown in the figure 4.1. This, being a simple system, will help in easy understanding of the entire problem and its solution. Whenever a general system differs from this example, the difference, and how that situation is handled in our approach, is explained side by side.

4.1 Input for the System

As stated earlier, this thesis attempts the static equilibrium analysis of the user defined system. Hence, here the input is provided by the user. On the basis of this input, the system is "constructed" by the code, is assembled and analyzed to get the equilibrium configuration.

The set of input required for describing the complete mechanical system consists of

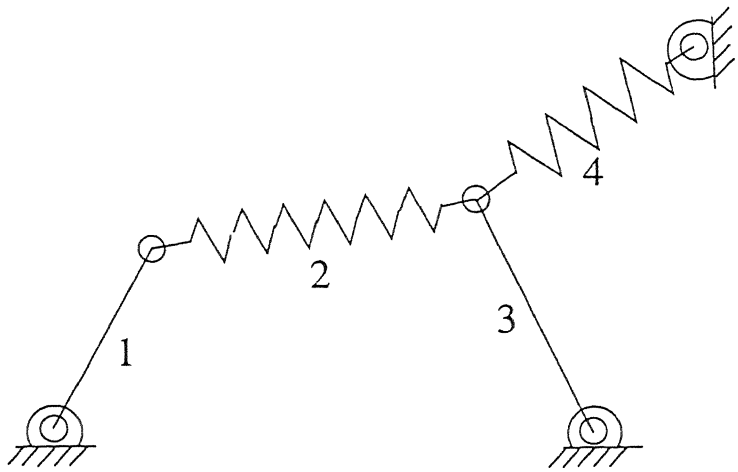


Figure 4.1: Mechanical system with 2 deformable members

1. Nature of the system (planar or spatial),
2. Total number of links in the system,
3. Number of deformable members,
4. Types of the deformable members (torsional, linear) and their corresponding stiffnesses,
5. Connectivity of all the links,
6. Dimensions of links, in terms of the locations of all joints of every link from the c.g. of the corresponding link, in the respective local frames,
7. Locations of the ground connections of the system,
8. External loadings (forces, moments) and their locations and orientations,
9. Weights of all the links,
10. Approximate values of the relative orientations of the links with respect to previous links along the loop.

The main advantage of asking the link geometry in terms of locations of all the joints with respect to c.g. in the local frame, is that the analysis remains perfectly suitable not only if all the links are binary, but also for the links with any number of joints.

The order in which this input is provided by the user, is given in the appendix, for a simple problem. It is important here, to note that the user has the freedom to enter approximate values of g.c. only. The other input parameters are to be provided, as accurately as possible.

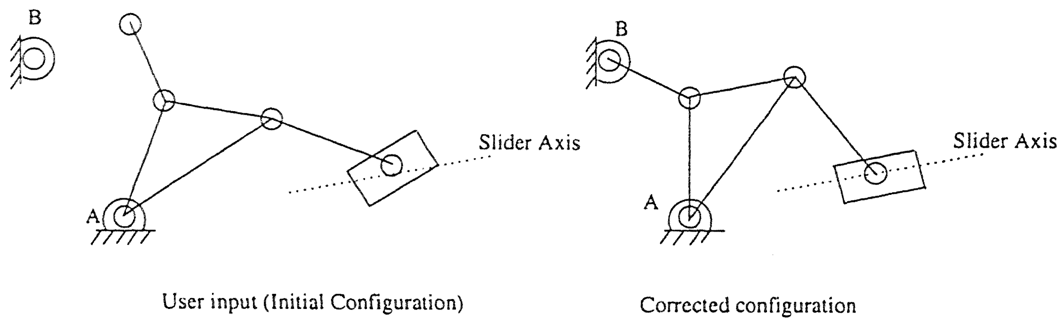


Figure 4.2: Correction of user input during assembly

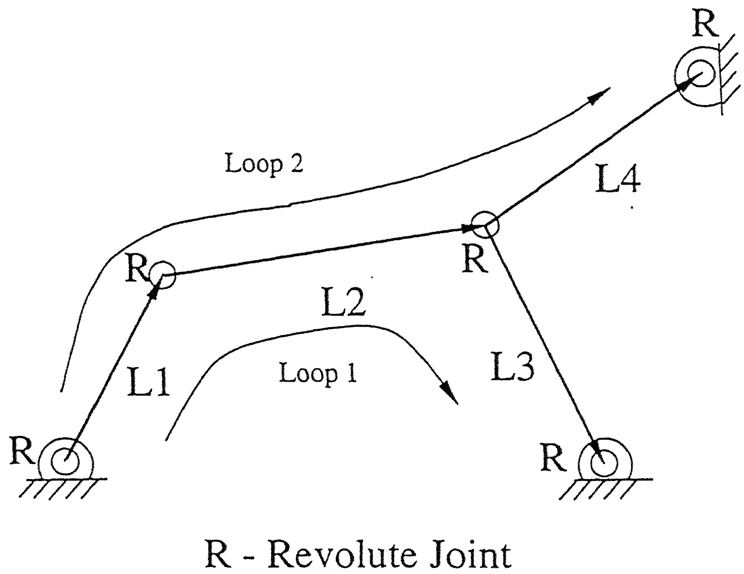


Figure 4.3: Kinematic loops are found out.

4.2 Assembly Procedure

As stated earlier, providing the correct initial relative orientations, for the complex planar and spatial mechanical systems is a tough task for the user. To help him through this, the assembly part is used, in which using the optimization techniques, we correct the relative orientations and get the initial correct configuration for the system.This is as shown in the figure 4.2.

The procedure for assembly is as follows. During the assembly, all the links of the structure are assumed to be rigid, i.e. in this case, links 2 and 3 are also considered to be rigid. All the loops in the structure, starting from the global origin are found out. These loops will give us the constraint equations as explained using figure 4.3.

Here, there are 2 loops, loop 1 with connectivity 1-2-3, and loop 2 with connectivity 1-2-4. Along the loop 1, link 1 is connected to the ground with revolute joint, hence ϕ_1 i.e. the relative angle between the link 1 and base, in the reference system of link 0 (ground), is the

generalized coordinate associated with link 1. Being a revolute joint, only one orientation is sufficient to define link 1 with respect to the base. Similarly, the joint between 1 and 2 is also a revolute joint. Hence only 1 g.c. is associated with link 2 viz. ϕ_2 , the relative angle between the links 1 and 2. Similarly, link 3 will have ϕ_3, the relative angle between link 2 and 3, as the only g.c associated with it.

In the second loop 1-2-4, all the connections are revolute joints. Generalized coordinates of links 1 and 2, are already defined as ϕ_1 and ϕ_2. Hence only the g.c. of link 4 is remaining which is ϕ_4, the relative angle between the links 2 and 4. Thus, now every link is having g.c. associated with it and the total number of g.c. for the whole structure is 4 (i.e. $\phi_1, \phi_2, \phi_3, \phi_4$).

The number of g.c for every link, depends upon the type of joint with which the link is connected with the previous link, along a particular loop. For example, if the connection between link 1 and link 2, in this case, were a spherical joint, then link 2 would have 3 g.c., the Euler angles of link 2 with respect to link 1.

Thus, the type of joints along a loop, influence the number of g.c. for every link in the loop. The g.c. for the links connected by various joints, with their previous links, are as given below:

- Revolute joint : 1 (one orientation with respect to previous link),
- Prismatic joint : 1(one displacement with respect to previous link),
- Spherical joint : 3 (3 Euler angles with respect to previous link),
- Cylindrical joint : 2 (one orientation, one displacement with respect to previous link) etc.

4.2.1 Redundant g.c.

In some cases, there is a possibility of presence of redundant g.c. For example, consider a binary link connected to the adjoining links with two spherical joints. In this case, the link should have three g.c., the three Euler angles with respect to previous link along the loop. But this link has a redundant freedom of rotation about the axis, connecting the joints, which is independent of the system as a whole. Hence, the third g.c. of the link cannot be determined by the force equilibrium method. In such cases, only the first two g.c. for the link i.e. the orientations about the z and y axes are considered during the analysis.

Similar situation also occurs, while considering the link with spherical-cylindrical joint. Heuristic approach is applied to detect such cases and only the dependent g.c. are used for analysis.

4.3 Getting the Constraint Equations

For getting the constraint equations, loop closure equations are used, which were explained in the previous chapter.

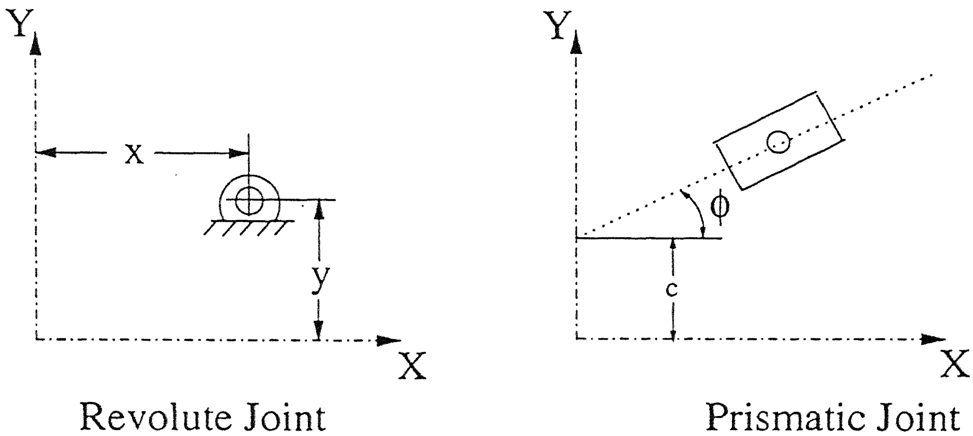


Figure 4.4: End constraints due to revolute and prismatic joints(planar)

4.3.1 Static determinacy check

This is the first step of the whole procedure. This check is important, as we are dealing only with the statically determinate systems. In this step, the information about the connectivity of the links, types of joints, deformable members, provided by the user is used. The method used for checking the static determinacy of user-defined system is explained earlier in the section 2.5. If the system is statically indeterminate, then the analysis is stopped and the case is informed to the user.

4.3.2 Number of constraint equations in each loop

4.3.2.1 Planar systems

In this approach, the constraints along every loop, depend upon the type of joint at the end of the loop, e.g. in a planar system, if a revolute joint is present at the end of the loop, it creates two constraints, viz. the global x and y coordinates of the end link, at that joint, are the same as the global x and y coordinates of the ground connection. In the case of a prismatic joint, the two end constraints are, the fixed orientation of the slider and its position along a fixed straight line. Both the end constraints are as shown in the figure 4.4.

4.3.2.2 Spatial systems

In the spatial systems, the number of constraints per loop are three or more depending upon the type of joint at the end of the loop. For a spherical joint at end, we get three constraints: the global x, y and z coordinates of the end link must be same as the global x, y, z coordinates of the corresponding ground connection.

A revolute joint has only one DOF, the orientation about the rotational axis. The other two orientations are constrained. Thus it offers, in all, five constraints, three position constraints and two orientation constraints. A prismatic joint gives five constraints, three orientation constraints and two position constraints, as the slider moves along a fixed direction.

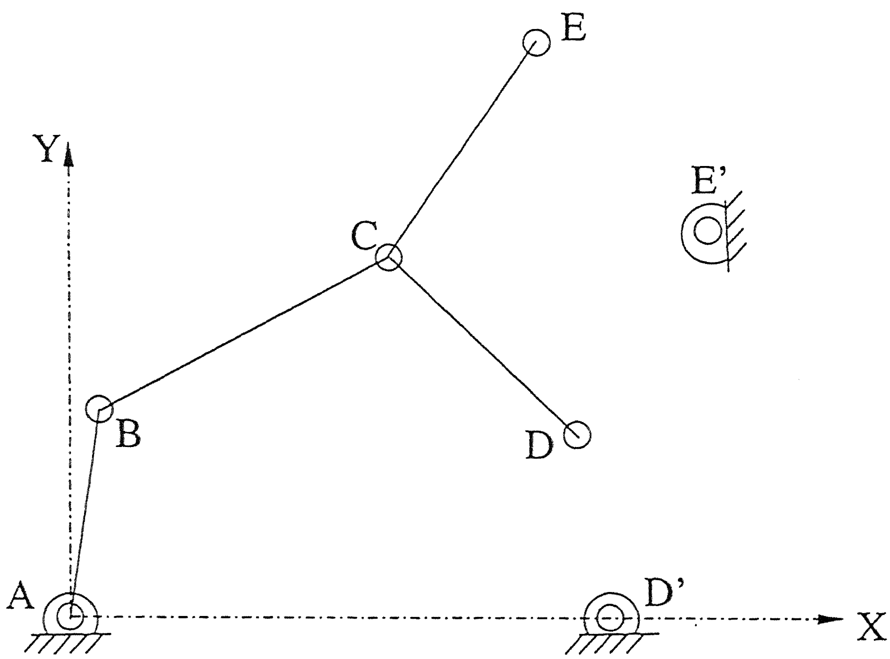


Figure 4.5: Approximate initial configuration provided by the user

4.3.3 Constraint equations for planar systems

The user provides the approximate values of all g.c. of the system. Thus, the values of $\phi_1, \phi_2, \phi_3, \phi_4$ in this case, are provided approximately. The connectivity of all the links is stored loop-wise in the linked-list. Since, the system is constructed along the loop, using the approximate values of the g.c, the approximately assembled structure will look like one shown in the figure 4.5, with joints D and E not coincident with the ground connections D' and E'. This will be corrected using the assembly part.

4.3.3.1 Revolute joint at the end of loop

For both the loops, in this example, the joints at ends are revolute. Hence, the constraints are that the points D and D' in loop 1 and E and E' in loop 2 must coincide for the feasibility of the system, i.e. the x and y coordinates of these points, must be same. For the planar systems, the z-coordinate is zero, but in spatial systems, the z-coordinate will also contribute to the constraints.

Using the loop closure equations, the location of joints D and E in the global reference frame, along the loops 1 and 2 respectively are calculated as

$$\{D\} = \left\{ \begin{array}{c} D_x \\ D_y \\ D_z \\ 1 \end{array} \right\} = T(0, \phi_1) * T(L_1, \phi_2) * T(L_2, \phi_3) * \{L_3\}$$

and

$$\{E\} = \left\{ \begin{array}{c} E_x \\ E_y \\ E_z \\ 1 \end{array} \right\} = T(0, \phi_1) * T(L_1, \phi_2) * T(L_2, \phi_4) * \{L_4\}$$

For the configuration to be feasible, the points D, D' and E, E' must be coincident (as links 3 and 4 are connected to the base at D and E respectively).

Thus, we have,

$$\{D\} = \left\{D'\right\} \text{ and } \{E\} = \left\{E'\right\}$$

i.e.

$$\left\{ \begin{array}{c} D_x \\ D_y \\ D_z \\ 1 \end{array} \right\} - \left\{ \begin{array}{c} D'_x \\ D'_y \\ D'_z \\ 1 \end{array} \right\} = \{0\} \quad and \quad \left\{ \begin{array}{c} E_x \\ E_y \\ E_z \\ 1 \end{array} \right\} - \left\{ \begin{array}{c} E'_x \\ E'_y \\ E'_z \\ 1 \end{array} \right\} = \{0\}$$

$\left\{D'\right\}$ and $\left\{E'\right\}$ are known, as the accurate coordinates of the ground connections at D and E are provided by user. In case of planar systems, z-coordinate being zero, we have

$$D'_z = E'_z = D_z = E_z = 0$$

Thus, we get 2 constraint equations per loop for the system, which are

$$f_1 = D_x - D'_x = 0;$$

$$f_2 = D_y - D'_y = 0;$$

$$f_3 = E_x - E'_x = 0;$$

$$f_4 = E_y - E'_y = 0.$$

Or,

$$\left\{ \begin{array}{c} f_1 \\ f_2 \end{array} \right\} = T(\phi_1) * T(L_1, \phi_2) * T(L_2, \phi_3) * \{L_3\} - \left\{ \begin{array}{c} D'_x \\ D'_y \end{array} \right\}$$

and

$$\left\{ \begin{array}{c} f_3 \\ f_4 \end{array} \right\} = T(\phi_1) * T(L_1, \phi_2) * T(L_2, \phi_4) * \{L_4\} - \left\{ \begin{array}{c} E'_x \\ E'_y \end{array} \right\}$$

In this way, four constraint equations are obtained for this planar structure (2 per loop).

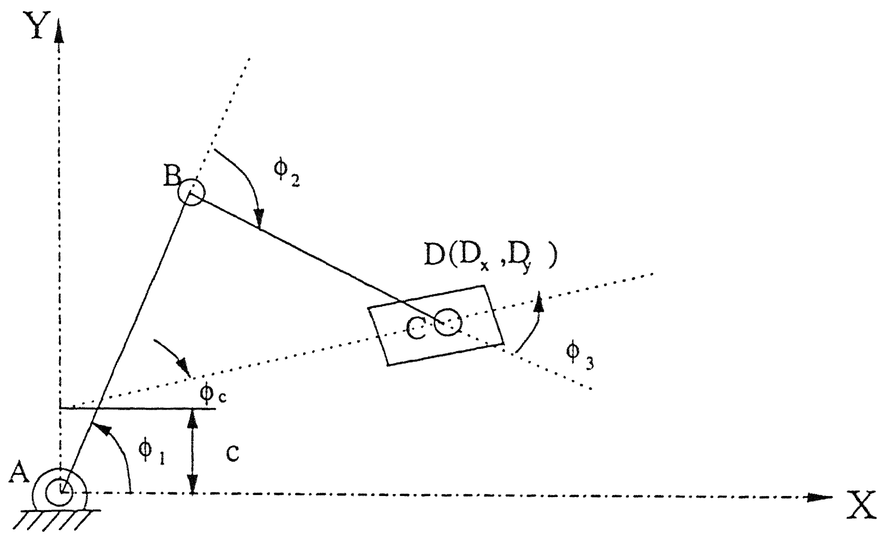


Figure 4.6: Prismatic joint at the end of loop

4.3.3.2 Prismatic joint at the end of loop

As explained in section 4.3.2, the two constraints due to prismatic joint at the end of loop are that the slider moves along a fixed straight line and its orientation with respect to z axis of global reference system is fixed. That is, we have

$$y - mx - c = 0 \quad \text{and} \quad \phi - \phi_c = 0$$

Thus, in figure 4.6, the two constraints obtained for the loop are

$$D_y - mD_x - c = 0 \tag{4.1}$$

and

$$\phi_1 + \phi_2 + \phi_3 - \phi_c = 0 \tag{4.2}$$

4.3.4 Constraint equations for spatial systems

In spatial systems, getting the governing constraint equations is a bit complicated. In this thesis, we have considered only the systems with spherical and revolute joints at the end of the loop. But this approach can be easily extended to other kind of joints like prismatic joint, cylindrical joint, screw joint etc.

4.3.4.1 Spherical joint at the end

The spherical joint offers three constraints as explained in section 4.3.2. The position of ground connection $D(D_x, D_y, D_z)$ is provided by the user (refer figure 4.7). The position D' of the last link of the loop is calculated using the loop closure equations. Hence, the three constraints are obtained as

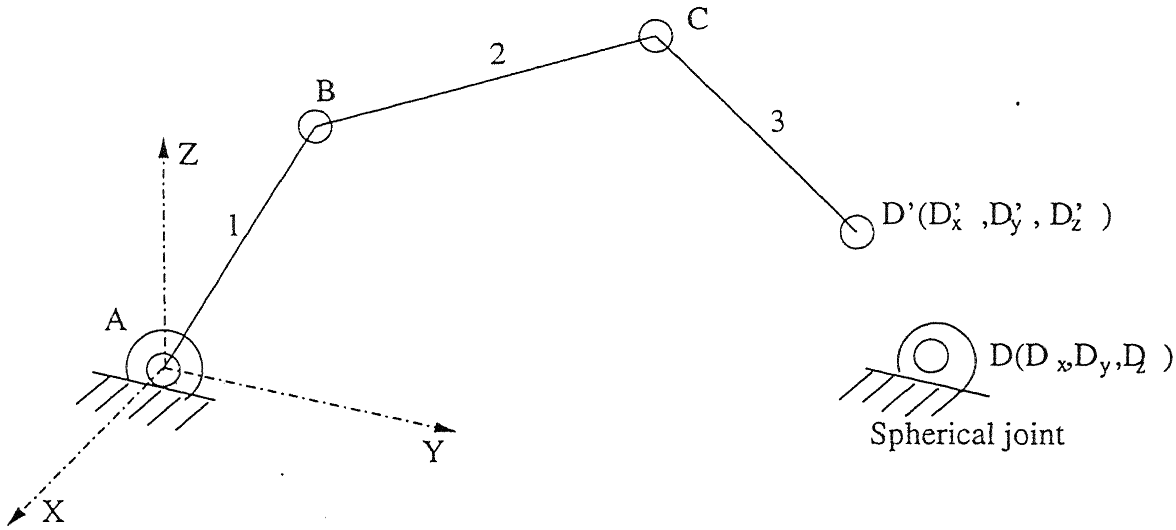


Figure 4.7: Spherical joint at the end of a loop

$$\begin{Bmatrix} f_1 \\ f_2 \\ f_3 \end{Bmatrix} = \begin{Bmatrix} D_x^{'} \\ D_x^{'} \\ D_x^{'} \end{Bmatrix} - \begin{Bmatrix} D_x \\ D_x \\ D_x \end{Bmatrix}$$

4.3.4.2 Revolute joint at the end

If a revolute joint is present at the end of the loop of a spatial system, then it offers five constraints. Of the five constraints, three constraints are the x, y, z position constraints. These are obtained in the same fashion as that of the spherical joint. The remaining two constraints are the orientation constraints. These are obtained as explained below.

In the figure 4.7, let us consider the joint D as the revolute joint. The homogeneous transformation up to joint C can be described as

$$T = \begin{bmatrix} \widehat{x_1} & \widehat{y_1} & \widehat{z_1} & \widehat{t} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

where $\begin{bmatrix} \widehat{x_1} & \widehat{y_1} & \widehat{z_1} \end{bmatrix}$ represents the rotation transformation in the initial frame and $\begin{bmatrix} \widehat{t} \end{bmatrix}$ is the position of joint C with respect to the global reference system. The user provides the direction cosines of the axis of revolution of the joint D. Let this be $\widehat{k} = \begin{bmatrix} l & m & n \end{bmatrix}^T$. The constraints here are that the direction cosines of the joint D' and $\widehat{k}$ should be same. Thus, the orientation of the axis of D' with respect to the local reference frame of the link 3 is given by

$$\widehat{d} = \begin{bmatrix} \widehat{x_1} & \widehat{y_1} & \widehat{z_1} \end{bmatrix}^T \widehat{k}$$

Hence the direction cosines of the joint D' in global reference should be same. Thus, we have

$$\begin{bmatrix} \widehat{x_2} & \widehat{y_2} & \widehat{z_2} \end{bmatrix} \hat{d} = \hat{k}$$

or

$$\begin{bmatrix} \widehat{x_2} & \widehat{y_2} & \widehat{z_2} \end{bmatrix} \begin{bmatrix} \widehat{x_1} & \widehat{y_1} & \widehat{z_1} \end{bmatrix}^T \hat{k} - \hat{k} = \hat{0}$$

where $\begin{bmatrix} \widehat{x_2} & \widehat{y_2} & \widehat{z_2} \end{bmatrix}$ is the rotation transformation in the changing frame (during the assembly and deformation).

This gives three constraint equations which are the functions of the g.c. But one of these constraints is dependent on the other two. Hence any of the two equations can be used as the constraint equations. We have used the first two terms of the vector as the orientation constraints.

A library of various transformation matrices, for all types of joints, is created for accomodating any type of kinematic joint between the members of the structure. To this library, when the values of g.c. and link lengths (link length is a scalar representing length of the link between the joints along the loop) are provided, the homogeneous transformation matrices are returned depending upon the type of joint. The matrices are multiplied in order, along the loop, so as to get the constraint equations.

Thus, we get the number of constraint equations same as the number of g.c. These equations are solved, to get the correct initial configuration in terms of the g.c.

4.4 Getting Jacobian (∇f)

4.4.1 Jacobian

Jacobian of the constraint equations is essential to check the singular behavior of the system and is also required in the analysis procedure.

We have constraint equations as

$$f(X) = 0\,;$$

so the Jacobian is

$$\nabla f = \frac{\partial f}{\partial X} = \begin{bmatrix} \frac{\partial f_1}{\partial \phi_1} & \frac{\partial f_1}{\partial \phi 2} & - & \frac{\partial f_1}{\partial \phi_n} \\ - & - & - & - \\ - & - & - & - \\ \frac{\partial f_n}{\partial \phi_1} & \frac{\partial f_m}{\partial \phi_2} & - & \frac{\partial f_m}{\partial \phi_n} \end{bmatrix}$$

The calculation of Jacobian, requires the partial derivatives of the constraint equations, with respect to every g.c. Here, writing the constraint equations in terms of loop closure equations, becomes helpful. The method to get the partial derivatives of the constraint equations for the planar and spatial systems is explained below.

4.4.1.1 Planar systems

In the planar example used for explanation, we have revolute joints at end. For loop 1, we have

$$\left\{ \begin{array}{c} f_1 \\ f_2 \end{array} \right\} = T(\phi_1) * T(L_1, \phi_2) * T(L_2, \phi_3) * \{L_3\} - \left\{ \begin{array}{c} D_x' \\ D_y' \end{array} \right\}$$

The partial derivatives of f_1 and f_2 with respect to a g.c., say, ϕ_1 , i.e. $\frac{\partial f_1}{\partial \phi_1}$ and $\frac{\partial f_2}{\partial \phi_1}$ are calculated as

$$\left\{ \begin{array}{c} \frac{\partial f_1}{\partial \phi_1} \\ \frac{\partial f_2}{\partial \phi_1} \end{array} \right\} = T'(\phi_1) * T(L_1, \phi_2) * T(L_2, \phi_3) * \{L_3\}$$

similarly, partial derivative with respect to ϕ_2 are

$$\left\{ \begin{array}{c} \frac{\partial f_1}{\partial \phi_2} \\ \frac{\partial f_2}{\partial \phi_2} \end{array} \right\} = T(\phi_1) * T'(L_1, \phi_2) * T(L_2, \phi_3) * \{L_3\}$$

In this way, the partial derivatives of the constraint equations in loop 1, with respect to all the g.c, are calculated.

In the case of prismatic joint, the partial derivatives of the equations 4.1 and 4.2 with respect to the g.c., are to be used in the Jacobian.

4.4.1.2 Spatial systems

In the case of a spherical joint, the procedure is same as that of the planar revolute joint, except that the third constraint i.e. the z-position constraint is also partially differentiated.

The position constraints of the revolute joints are differentiated in the same way as that of the spherical joints. The orientation constraints are complex in nature and differentiation explained in chapter three does not work. Hence these constraints are to be numerically differentiated. We have used the Newton's forward difference method from Sastry [9] for the numerical differentiation.

Thus, every constraint equation is partially differentiated with respect to the g.c. and we finally get the Jacobian of all rigid constraints.

4.5 Detection of singularity

When the Jacobian is rank deficient, i.e. its determinant is zero, the initial configuration provided by the user, is itself singular. That is, if

$$\left| \frac{\partial f}{\partial X} \right| = 0 \, ,$$

then the configuration is singular.

In such a case, the user is informed to alter the approximate input values of the g.c., so as to get the correct initial configuration.

But the main use of this Jacobian, is to detect, the singularity of the configuration after assembly. If the assembled configuration is singular, it will have unconstrained mobility and hence it cannot attain a stable equilibrium configuration. In this situation, the user is informed of the singularity and the further analysis is stopped.

4.6 Getting Hessian ($\nabla^2 f_k$) of Constraint Equations

The Hessian matrix of the constraint equations is useful in the analysis procedure. It is calculated in the similar fashion as the Jacobian.

The Hessian is given by

$$\nabla^2 f_k = \frac{\partial^2 f_k}{\partial \phi_i \partial \phi_j}$$

The second order partial derivatives of f_1 and f_2 with respect to any g.c., say ϕ_1 are calculated as

$$\left\{ \begin{array}{c} \frac{\partial^2 f_1}{\partial^2 \phi_1} \\ \frac{\partial^2 f_2}{\partial^2 \phi_1} \end{array} \right\} = T''(\phi_1) * T(L_1, \phi_2) * T(L_2, \phi_3) * \{L_3\}$$

Similarly, the second order partial derivatives of f_1 and f_2 with respect to ϕ_1 and ϕ_2 are calculated as

$$\left\{ \begin{array}{c} \frac{\partial^2 f_1}{\partial \phi_1 \partial \phi_2} \\ \frac{\partial^2 f_2}{\partial \phi_1 \partial \phi_2} \end{array} \right\} = T'(\phi_1) * T'(L_1, \phi_2) * T(L_2, \phi_3) * \{L_3\}$$

In this way, all the terms of the the Hessian matrix $\nabla^2 f_k$ for each constraint equation are calculated. For the planar slider at the end, the second order partial derivatives of the equations 4.1 and 4.2 are used. In case of the spatial revolute joint at the end, the numerical differentiation is used.

In this way, the Hessian matrix of each constraint equations is calculated.

4.7 Optimization Technique

The constraint equations obtained, are a set of nonlinear equations. The solution of these equations, will give us the correct initial configuration. Either, we can solve the set of simultaneous equations, i.e.

$$\left\{ \begin{array}{c} f_1 \\ f_2 \\ - \\ f_n \end{array} \right\} = \{0\}$$

or we can pose the problem in such a way that the sum of squares of all the constraint equations is zero, i.e.

$$f_1^2 + f_2^2 + f_3^2 + - - - - - + f_n^2 = 0; \tag{4.3}$$

Equation 4.3 can be looked as optimization problem with

$$U = f_1^2 + f_2^2 + f_3^2 + - - - - - + f_n^2; \tag{4.4}$$

with U as the objective function to be minimized.

Thus, it becomes an unconstrained nonlinear multi-variable optimization problem.

4.7.1 Selection of optimization technique

Various methods that can be used for this optimization problem are Newton-Raphson method, Cauchy's method, Levenberg-Marquardt method etc. Newton's method has two drawbacks. This method works well only when the initial guess (the g.c. in this case) is close to the correct value, i.e. its local search is good, but global search is bad. Also, if the configuration is singular, this method fails. Cauchy's method is a global search method and hence even if the user input is away from the correct value, it converges to the correct value. But it suffers from the drawback that, at the vicinity of the correct solution, its convergence is slow. As far as the initial guess given by the user is concerned, it is usually not known whether it is close or away from the optimum. Hence, to solve this problem, Levenberg- Marquardt method is used as it takes the advantage of both Cauchy and Newton method.

In the Levenberg-Marquardt method, Cauchy's method is initially followed. Thereafter, Newton's method is adopted. Hence initially the solution approaches the correct value irrespective of the initial guess given by the user and finally it converges fast due to the use of Newton's method, the convergence is fast. As a result, the overall convergence of the method is good. The algorithm of the method is taken from Deb [10].

ALGORITHM :

Step 1: Take the initial configuration, provided by the user, as the starting point, X_0.
Choose the maximum number of iterations, M and a termination parameter, ϵ .
Set $k = 0$ and $\lambda_0 = 10^4$ (a large number).
Step 2: Calculate $\nabla U(X_k)$.
Step 3: If $\|\nabla U(X_k)\| \leq \epsilon$ or $k \geq M$? **Terminate.**
Step 4: Calculate $s(X_k) = -\left[\nabla^2 U + \lambda_k I\right]^{-1} \nabla U(X_k)$. Set $X_{k+1} = X_k + s(X_k)$.
Step 5: Is $U(X_{k+1}) < U(X_k)$? If yes, go to Step 6.
Else go to step 7.
Step 6: Set $\lambda_{k+1} = \frac{1}{2}\lambda_k$, $k = k + 1$, go to Step 2.
Step 7: Set $\lambda_k = 2\lambda_k$ and go to Step 4.

A large value of the parameter λ is used initially. Thus, the Hessian matrix $\nabla^2 U$, has little effect on the determination of the search direction and the search is similar to Cauchy's method. After, a number of iterations, the value of λ becomes small and the effect is more like the Newton's method.

This method requires the gradient ∇U and Hessian $\nabla^2 U$ of the objective function. From equation (4.4), we have

$$\nabla U = 2f_1 \nabla f_1 + 2f_2 \nabla f_2 + ------ + 2f_n \nabla f_n$$

$$= 2\sum_i f_i \nabla f_i$$

Differentiating,

$$\nabla^2 U = 2\left[\sum_i \nabla f_i \nabla f_i^T + \sum_i f_i \nabla^2 f_i\right]$$

Thus, the optimum value of X i.e. the set of g.c. is obtained. This gives the correct initial configuration of the system. This configuration is the starting point for the static equilibrium analysis.

Chapter 5

Static Equilibrium Analysis

In the second phase of the analysis, the deformation of the deformable members is considered. The lengths of the deformable linear springs and angles of the torsional springs are considered as deformable coordinates. Here, the lengths of links 2 and 4 i.e. L_2 and L_4 are the deformable coordinates.

These deformable coordinates (d.c.) θ, are expressed as a function of the g.c., i.e.

$$\theta = g(X)$$

as explained below.

These deformable coordinate equations will replace some of the rigid constraint equations and those resulting equations (set of rigid constraints and d.c.) will be finally used in the optimization.

5.1 Procedure to get the Deformable Coordinates $g(X)$:

Here, we have two deformable coordinates, L_2 in the loop 1 , and L_4 in the loop 2.

5.1.1 Linear deformable link:

Both the deformable links are linear. Of the two, consider the deformable link BC.

Let the point B be at (x_1, y_1) and point C at (x_2, y_2) with respect to the global reference system, as shown in the figure 5.1. The deformable coordinate L_2 is equal to the distance between the points B and C. Thus,

$$g_1(X) = l(BC)$$

i.e.

$$g_1(X) = \sqrt{x^2 + y^2}$$

where $\sqrt{x^2 + y^2}$ is the distance between the joints B and C, approached from both ground connections of the loop in which the flexible member is present. Here,

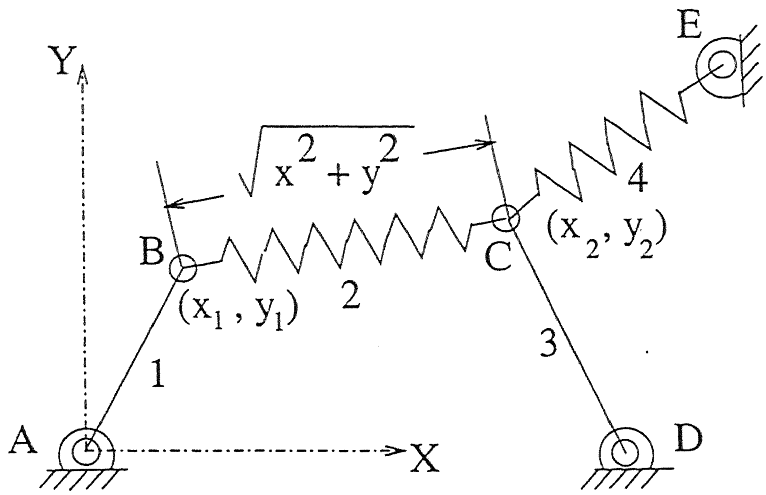


Figure 5.1: Deformable coordinate

$$x = x_2 - x_1 \tag{5.1}$$

and

$$y = y_2 - y_1 \tag{5.2}$$

and the constraint due to deformable coordinate 1 is

$$g_1(X) - \theta_1 = 0$$

i.e.

$$g_1(X) - L_2 = 0$$

The set of x_1, y_1, x_2, y_2 are calculated along the loop in which the d.c. is present (i.e. loop 1, in this case)

Using the loop closure equation, we get

$$\left\{ \begin{array}{c} x_1 \\ y_1 \end{array} \right\} = T(\phi_1) * \{L_1\}$$

and

$$\left\{ \begin{array}{c} x_2 \\ y_2 \end{array} \right\} = \left\{ \begin{array}{c} D_x \\ D_y \end{array} \right\} - T(\phi_1) * T(\phi_2) * T(\phi_3) * \{L_3\}$$

Putting these values of x_1, y_1, x_2, y_2 in equations (5.1) and (5.2) we get x and y, as the functions of the g.c. Similarly, the second deformable coordinate $g_2(X)$ can be calculated, giving the deformable constraint equation as

$$g_2(X) - L_4 = 0$$

In this way, for every deformable coordinate in the system, set of x and y is calculated as a function of g.c. The calculation of x_1, y_1, x_2, y_2 is same irrespective of the type of the joint in the loop. The only difference it makes is the concatenation of homogeneous transformation matrices changes depending upon the joints along the loop.

Thus we get, all the deformable coordinate equations as a function of the g.c.

5.1.2 Deformable links at the start and end of the loop

In case a deformable member is at the start of the loop, that means the link is connected to the ground at the global origin. Hence we have, $(x_1, y_1) = (0, 0)$.

Similarly, if the deformable member is at the end of the loop, the coordinates (x_2, y_2) are the same as the coordinates of the end ground connection of the loop.

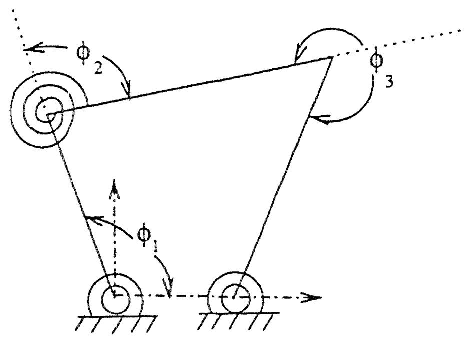


Figure 5.2: Structure with torsion spring as deformable member

5.1.3 Torsional deformable members

An example of structure with a torsional member is as shown in figure 5.2.

Here, $X = [\phi_1, \phi_2, \phi_3]$ are the g.c. There is only 1 loop and hence only two rigid constraint equations are available. The deformable coordinate here, is ϕ_c, the torsion spring angle. Thus, the deformable coordinate in this case, is equal to the difference between the relative angles ϕ_1 and ϕ_2.

Thus, the constraint equation for torsion spring is taken as

$$\phi_2 - \phi_1 = \phi_c;$$

i.e.

$$\phi_2 - \phi_1 - \phi_c = 0$$

In general, the deformable coordinate equation for torsional element is same as that used in the assembly part, i.e.

$$\phi_j - \phi_i - \phi_c = 0 \tag{5.3}$$

Note: In the case of assembly of mechanical system with torsion springs, the same equation (5.3) is used as the rigid constraint. Since the torsion spring is not in any loop, equation (5.3) acts as an additional constraint.

In this thesis, the analysis of spatial mechanical system with torsion spring elements is not considered.

5.2 Obtaining the Gradient of Deformable Coordinate (∇g)

The equilibrium equations contain a term $\frac{\partial g}{\partial X}$. This $\frac{\partial g}{\partial X}$is calculated as below. we know that, for every linear deformable coordinate,

$$g(X) = \sqrt{x^2 + y^2}$$

Hence,

$$\nabla g = \frac{\partial g}{\partial X} = (x^2+y^2)^{-\frac{1}{2}} * (x\nabla x + y\nabla y) \tag{5.4}$$

we also have,

$$\left\{\begin{array}{c} x \\ y \end{array}\right\} = \left\{\begin{array}{c} x_2 \\ y_2 \end{array}\right\} - \left\{\begin{array}{c} x_1 \\ y_1 \end{array}\right\} = \left\{\begin{array}{c} D_x \\ D_y \end{array}\right\} - T(\phi_1) * T(\phi_2) * T(\phi_3) * \{L_3\} - T(\phi_1) * \{L_1\}$$

The ∇x and ∇y are given by

$$\nabla x = \left[\frac{\partial x}{\partial \phi_1}, \frac{\partial x}{\partial \phi_2}, - - - \frac{\partial x}{\partial \phi_n}\right]^T,$$

$$\nabla y = \left[\frac{\partial y}{\partial \phi_1}, \frac{\partial y}{\partial \phi_2}, - - - \frac{\partial y}{\partial \phi_n}\right]^T$$

partial differentiation with respect to ϕ_1, ϕ_2 gives

$$\left\{\begin{array}{c} \frac{\partial x}{\partial \phi_1} \\ \frac{\partial y}{\partial \phi_1} \end{array}\right\} = -T'(\phi_1) * T(\phi_2) * T(\phi_3) * \{L_3\} - T'(\phi_1)\{L_1\}$$

$$\left\{\begin{array}{c} \frac{\partial x}{\partial \phi_2} \\ \frac{\partial y}{\partial \phi_2} \end{array}\right\} = -T(\phi_1) * T'(\phi_2) * T(\phi_3) * \{L_3\} - \{0\}$$

In this way, ∇x and ∇y are calculated. These, when used in equation (5.4), give us the gradient ∇g.

Similar procedure is adopted for every deformable coordinate.

5.3 Getting $\nabla^2 g$:

Differentiating equation (5.4) further, we get

$$\nabla^2 g = -(x^2+y^2)^{-\frac{3}{2}}(x\nabla x + y\nabla y) * (x\nabla x + y\nabla y)^T$$

$$+(x^2+y^2)^{-\frac{1}{2}}(\nabla x \nabla x^T + x\nabla^2 x + \nabla y \nabla y^T + y\nabla^2 y)$$

This requires the values of $\nabla^2 x$ and $\nabla^2 y$, which as calculated as below:

$$\left\{\begin{array}{c} \frac{\partial^2 x}{\partial \phi_1^2} \\ \frac{\partial^2 y}{\partial \phi_1^2} \end{array}\right\} = -T''(\phi_1) * T(\phi_2) * T(\phi_3) * \{L_3\} - T''(\phi_1)\{L_1\}$$

$$\left\{\begin{array}{c} \frac{\partial^2 x}{\partial \phi_1 \partial \phi_2} \\ \frac{\partial^2 y}{\partial \phi_1 \partial \phi_2} \end{array}\right\} = -T'(\phi_1) * T'(\phi_2) * T(\phi_3) * \{L_3\} - \{0\}$$

etc. Hence using the values of $x, y, \nabla x, \nabla y, \nabla^2 x, \nabla^2 y$, the Hessian $\nabla^2 g$ is obtained for all deformable coordinates (link lengths 2 and 3, in this case)

5.4 Combining Constraints

Every loop gives two constraint equations for planar systems. If the loop has a deformable coordinate, then the deformable constraint is also present. But actually, any two constraints per loop for planar system, are sufficient to describe the system completely and hence the third constraint is redundant. Since we are interested in the deformations in deformable members, we can replace any of the rigid constraints, by the d.c. equation. Thus, the constraint equations include the d.c. and, in all, two constraints per loop are used. We have replaced the y-coordinate constraint with the deformable constraint in case of a revolute joint and the orientation constraint in case of a prismatic joint at the end of the loop.

In this case, the d.c being in loop 1 and loop 2 respectively, we have dropped the rigid constraint due to y- coordinate in each loop and replaced it by the corresponding d.c. equation. Hence, we have constraint equations as

$$f(X) = 0$$

$$g(X) - \theta = 0$$

i.e.

$$\left\{ \begin{array}{c} f_1 \\ f_3 \\ g_1 - L_2 \\ g_2 - L_3 \end{array} \right\} = \{0\}$$

as the constraints $f_2 = 0$ and $f_4 = 0$ are dropped.

In case of spatial systems, we have replaced the z-coordinate constraint in case of a spherical joint and one of the orientation constraints in case of revolute joint at the end of the loop.

If the loop contains more than one deformable coordinate, then the same number of rigid constraints in that loop are replaced by the the deformable coordinate equations. If the number of deformable coordinates in a loop are greater than two, then the first two coordinates are considered in the same loop and the remaining are considered as the members of some other loops in which also, these members are present.

When a constraint is dropped , its Jacobian and Hessian are also dropped and replaced by the Jacobian and Hessian of the d.c. in that loop.

If torsion spring is present in the system, since it is not a member of any loop, it does not replace any of the rigid constraints and simply gets added as one more constraint equation for the system.

In this way, the number of constraint equations still remain n, which help us get the g.c., every time we get the new values of deformable coordinates through the equilibrium equation.

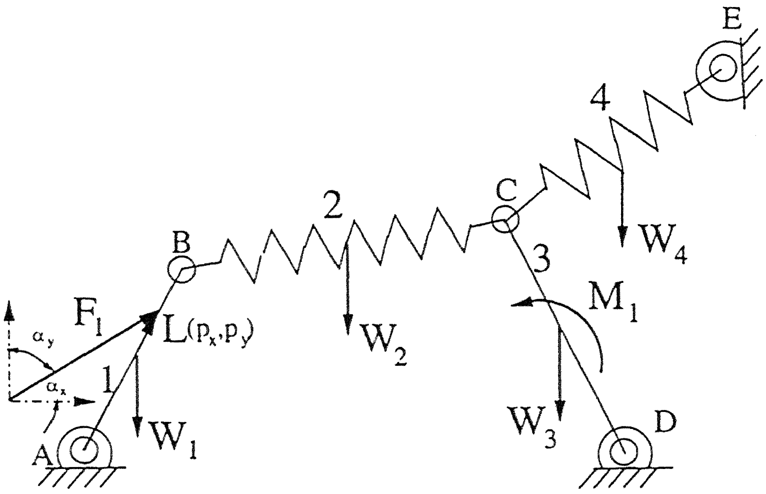


Figure 5.3: External loading on the system

5.5 Generalized Force (F_{load})

The system deforms under the action of external loading. These forces include external forces, torques, and body forces i.e. the weights of the links.

Consider a force F_1, acting on link 1 of the system, at angles α_x, α_y with respect to the x and y axis of the global reference system, at a position (p_x, p_y) from the c.g. of the link 1. Also, consider a moment M_1, acting counter-clockwise at the c.g. of link 3 of the system. Let W_1, W_2, W_3, W_4 be the weights of links 1, 2, 3 and 4 respectively as shown in figure 5.3.

From the information provided by the user, about the geometry of link 1 and the location of the force, with respect to c.g. of the link 1, the position of the force from joint A, in the global reference frame, is obtained. Let this be $L = [l_x, l_y, l_z, 1]^T$. Let (x, y) be the location of the force, with respect to the global reference system. This location (x, y) in terms of the g.c. is given by

$$\left\{ \begin{array}{c} x \\ y \end{array} \right\} = T(0, \phi_1) * \{L\} \tag{5.5}$$

Let $(x_1, y_1), (x_2, y_2), (x_3, y_3), (x_4, y_4)$ be the location of c.g. of every link in the global frame, and $\{C_1\}$,$\{C_2\}$,$\{C_3\}$ and $\{C_4\}$ are their local position vectors, from the previous joints, along the loop, then we have,

$$\left\{ \begin{array}{c} x_1 \\ y_1 \end{array} \right\} = T(0, \phi_1) * \{C_1\}$$

$$\left\{ \begin{array}{c} x_2 \\ y_2 \end{array} \right\} = T(0, \phi_1) * T(L_2, \phi_2) * \{C_3\}$$

$$\left\{ \begin{array}{c} x_3 \\ y_3 \end{array} \right\} = T(0, \phi_1) * T(L_1, \phi_2) * T(L_2, \phi_3) * \{C_3\}$$

$$\left\{ \begin{array}{c} x_4 \\ y_4 \end{array} \right\} = T(0, \phi_1) * T(L_1, \phi_2) * T(L_2, \phi_4) * \{C_4\}$$

The partial derivatives of equation (5.5), with respect to $\phi_1, \phi_2, \phi_3, \phi_4$ are given by

$$\left\{ \begin{array}{c} \frac{\partial x_1}{\partial \phi_1} \\ \frac{\partial y_1}{\partial \phi_1} \end{array} \right\} = T^{'}(\phi_1)\{L\}, \qquad \left\{ \begin{array}{c} \frac{\partial x_1}{\partial \phi_2} \\ \frac{\partial y_1}{\partial \phi_2} \end{array} \right\} = \{0\},$$

$$\left\{ \begin{array}{c} \frac{\partial x_1}{\partial \phi_3} \\ \frac{\partial y_1}{\partial \phi_3} \end{array} \right\} = \{0\}, \qquad \left\{ \begin{array}{c} \frac{\partial x_1}{\partial \phi_4} \\ \frac{\partial y_1}{\partial \phi_4} \end{array} \right\} = \{0\}$$

In case of moments, the relation between the angle of link on which the moment is acting, with respect to the global reference system and the g.c. is given by the algebraic sum of g.c. along the loop up to the link, at which the moment is acting. Thus in this case, we have,

$$\phi = \phi_1 + \phi_2 + \phi_3$$

therefore, we have,

$$\frac{\partial \phi}{\partial X} = [1, 1, 1, 0]^T$$

Spatial systems

In case of the spatial systems, the z component of the force also comes into picture. Hence, the partial derivatives of the equation for z-coordinate of the force is also to be taken into consideration.

In the case of moments input, the three components of the moments are to be entered. The relation between the three angles of the link, with respect to the global reference system and the g.c. is not directly available. This is calculated as explained below. Details can be found out in Craig [11]

Let the transformation matrix for the orientation of the link on which three components of moments are acting be

$$\left[\begin{array}{ccc} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{array} \right]$$

then, the three global angles are given by

$$\phi_x = Atan2(r_{32}, r_{33}),$$

$$\phi_y = Atan2\left(-r_{31}, \sqrt{r_{11}^2 + r_{21}^2}\right)$$

and

$$\phi_z = Atan2(r_{21}, r_{11})$$

where $Atan2(x, y)$ computes $tan^{-1}\left(\frac{y}{x}\right)$, but uses the signs of both x and y to determine the quadrant in which the resulting angle lies.

The partial differentiation of these angle with respect to the g.c is not straight forward and hence numerical differentiation was used.

Similarly, for the locations of all the forces and moments, the partial derivatives are calculated and we get the Jacobian J relating the derivatives of the locations of these forces with respect to that of g.c. The actual force acting on the system, is the set of force components, acting along the x, y directions, and the moments acting about the c.g. of some of the links. Here, the convention used is that, the force components acting in the direction global reference axes, are considered as positive. Similarly, the moments acting in the counterclockwise directions are considered as positive. For the given example of planar system, we have,

$$F_{actual} = [F_1 \cos \alpha_x, F_1 \cos \alpha_y, 0, -W_1, 0, -W_2, 0, -W_3, 0, -W_4, +M_1]^T$$

Thus, generalized force F_{load} is given by

$$F_{load} = J^T * F_{actual}$$

5.6 Equilibrium

The equation for static equilibrium is

$$F_{load} + \left\{ \left(\frac{\partial f}{\partial X}\right)^T \lambda + \left(\frac{\partial g}{\partial X}\right)^T F_\theta \right\} = 0$$

Re-arranging, we get,

$$\left\{ \frac{\partial f}{\partial X} \frac{\partial g}{\partial X} \right\}^T \left\{ \begin{array}{c} \lambda \\ F_\theta \end{array} \right\} = -F_{load}$$

we know F_{load}, $\frac{\partial f}{\partial X}$, $\frac{\partial g}{\partial X}$.

Thus, equilibrium equation is basically, a set of linear equations with λ and F_θ as unknowns. These are obtained by using the Gaussian elimination method.

Of λ and F_θ, F_θ is of much significance for us, as it gives the new set of deformable coordinates, after deformation.

The change in the deformable coordinates, i.e. $\Delta\theta$, is calculated from

$$F_\theta = [K] \, \Delta\theta$$

In this case, $\Delta\theta$ is ΔL_2, the deformation in link 2. From this ΔL_2, the new d.c. value is obtained as

$$(L_2)_{new} = L_2 - \Delta L_2$$

5.7 Iterative Procedure

With these new values of deformable coordinates, again the values of f, ∇f, $\nabla^2 f$, g, ∇g, $\nabla^2 g$ are obtained in the manner explained before. Again, the Levenberg-Marquardt method is applied to get the new configuration of the system, for those deformable coordinates.

5.7.1 Updating the F_{load}

As stated earlier, the positions of the external loadings (including weights), are considered fixed with respect to the global reference frame, during the whole analysis. Hence, after getting the new configuration after iteration 1, the locations of these forces, change with respect to the respective links, on which they are applied. Thus, the Jacobian relating the force locations and new generalized coordinates, also get changed. This new Jacobian is calculated as given below.

Consider the force F_1. Its location (x, y) is calculated with respect to the global reference system while calculating the F_{load} initially, as explained in section 5.5. Its location with respect to the link 1 is (p_x, p_y). After the first iteration, the system deforms and hence the location of the point with respect to the link 1 changes to $(p_x^{'}, p_y^{'})$, as shown in the figure 5.4. Due to this, the location of the point from joint A, in the local frame of link 1, changes. Let this be $L^{'} = \left[l_x^{'}, l_y^{'}, l_z^{'}, 1\right]$. Let the g.c. obtained, after iteration 1, be $(\phi_1^{'}, \phi_2^{'}, \phi_3^{'}, \phi_4^{'})$.

Thus the global position of the point is related to the new g.c. as,

$$T(0, \phi_1^{'}) * \left\{L^{'}\right\} = \left\{ \begin{array}{c} x \\ y \end{array} \right\}$$

This gives,

$$\left\{L^{'}\right\} = \left[T(0, \phi_1^{'})\right]^{-1} * \left\{ \begin{array}{c} x \\ y \end{array} \right\}$$

In this way, the new position vector of the point P, with respect to joint A, in local frame of link 1 is calculated.

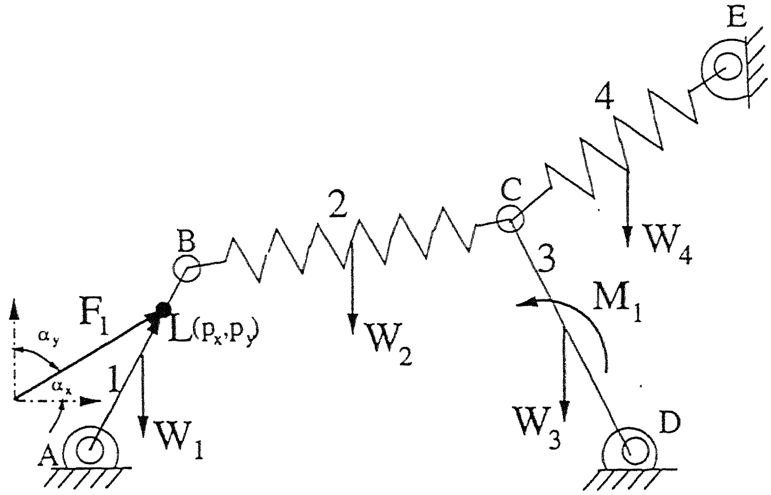
Thus, the position of point P can now be expressed in the global frame, using the new g.c. values as

$$\left\{ \begin{array}{c} x \\ y \end{array} \right\} = T(0, \phi_1^{'}) * \left\{L^{'}\right\}$$

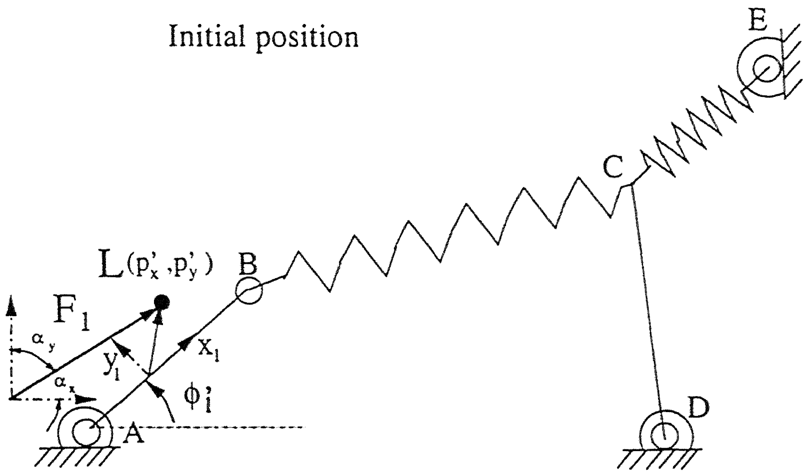
This equation is now used to get the new force Jacobian. This procedure is repeated for all the loading points and we get the new "updated" force Jacobian.

This new J, changes the F_{load}, as F_{actual} is constant. With new values of F_{load}, $\frac{\partial f}{\partial X}$, $\frac{\partial g}{\partial X}$, again the equilibrium equation is solved, to get new values of θ.

These iterations are repeated until θ converges. The set of generalized coordinates X, thus obtained, is the **final configuration** of the system, at the static equilibrium.



Initial position



Position after iteration 1

Figure 5.4: Position of the system initially and after iteration 1

Chapter 6

Results and Discussion

The static equilibrium analysis of a few planar and spatial mechanical systems is presented in this chapter. As discussed earlier, first the assembly is performed to get the initial correct configuration from the approximate user input. Then, the static equilibrium analysis is performed to get the final equilibrium position of the system. Numerical as well as graphical output of some examples are given. In the schematic figures, the various types of joints used in the examples are represented by: R-revolute, P-prismatic, S-spherical, SC- screw etc. In the case of assembly, the grey lines indicate the user input and black lines indicate the assembled configuration. In the case of equilibrium analysis, the black lines indicate the initial correct configuration and the grey lines indicate the deformed position.

Planar mechanical systems:

Example I: A five bar system is as shown in figure 6.1. It has two deformable members, link 2 and link 3. The other system parameters are as given below.
Link lengths: L1=L2=L3=L4=10 cm.
Spring stiffness: K2=K3=70 N/cm.
Position of ground connections: A(19.65,2.59,0), B(23.31,16.25,0)
Force data: F1=400N at an angle of -60^0 with x-axis at the c.g. of link 1.
Moment data: M1=100 N-cm acting in counter-clock-wise direction, at the c.g. of link 4.
User input of orientations $(\phi_1, \phi_2, \phi_3, \phi_4) = (50, -40, -70, 20)$

The assembled configuration is as shown in the figure 6.2. The initial correct configuration obtained after the assembly is $(59.95, -44.90, -75.50, 15.06)$. The exact values for the system are $(60, -45, -75, 15)$. The number of iterations required =16.

The deformed position of the system is as shown in the figure 6.3. The algorithm converges in 3 iterations giving the equilibrium configuration as $(49.89, -43.27, -68.64, 40.02)$. The deformable coordinates obtained are L1=10.03 cm and L2=6.90 cm.

Example II: This example shows a system with a ternary link and the remaining members as the deformable members (ref. figure 6.4). The parameters of the system are:
Link lengths: L1=L3=L4=10 cm. Lengths of the three sides of the ternary link are l(LM)=10 cm, l(LN)=l(MN)=11.18 cm.
Spring stiffness: K1=K3=K4=100N/cm.
Position of ground connections: A(27.32,0,0), B(8.66, 23.66, 0).

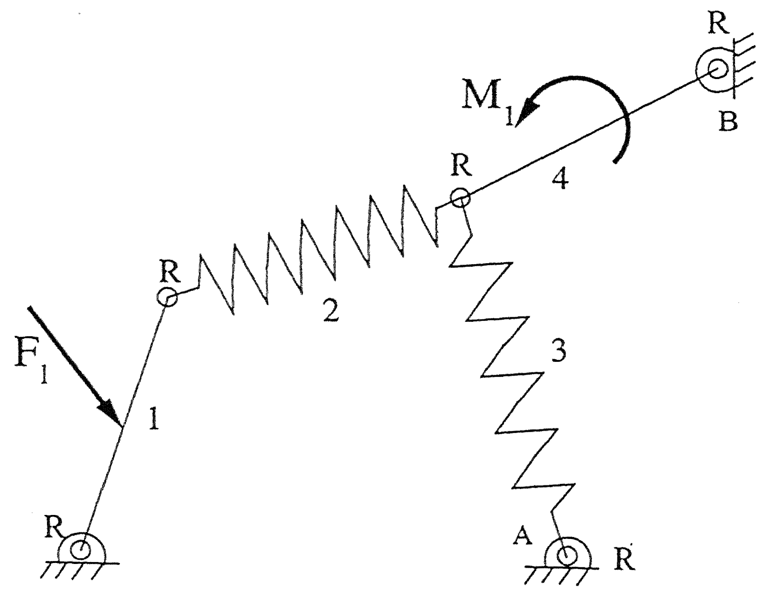


Figure 6.1: Example I

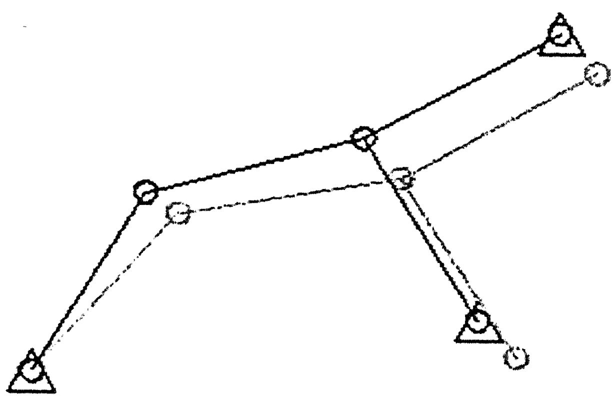


Figure 6.2: Assembly of Example I

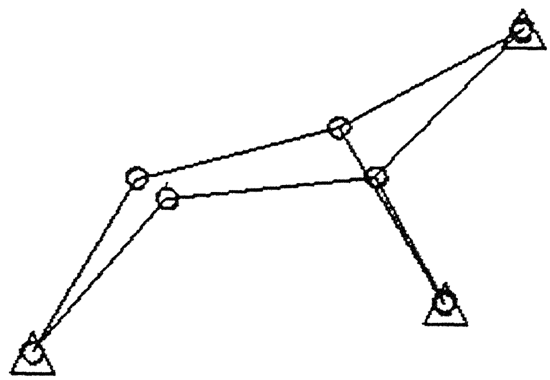


Figure 6.3: Static equilibrium of Example I

Force data: Force F1=300 N acting on link 2 at an angle of 60^0 with x-axis.
Moment data: Moment M1= 50N-cm acting in the clockwise direction, at the c.g. of link 2.
Weight of links: W1=W2=W3=W4=2N.
User input of orientations $(\phi_1, \phi_2, \phi_3, \phi_4) = (20, -35, -10, 100)$.

The assembled configuration is as shown in the figure 6.5. The correct initial configuration obtained is $(29.45, -29.10, -30.25, 119.43)$ after 53 iterations. The exact values are $(30, -30, -30, 120)$.

Under the action of the loading, the system deforms to give the final configuration as $(30.57, -25.20, -39.94, 110.80)$. The deformable coordinates obtained are L1 = 10.01 cm, L2 = 10.44 cm and L3 = 8.92 cm (ref. figure 6.6). The number of iterations required = 6.

Example III: This example shows a structure with a slotted lever and two deformable links (figure 6.7). The parameters of the system are:
Link lengths: L1=L4=10 cm. L3=5 cm.
Spring stiffness: K3=K4=100N/cm.
Position of ground connections: A(5,0,0), B(13.66, 3.66, 0).
Force data: Force Fl = 300 N acting on link1 at an angle of 120^0 with x-axis.
Moment data: Moment M1= 50N-cm acting in the counter clockwise direction, at the c.g. of link 1.
User input of orientations $(\phi_1, \phi_2, \phi_3, \phi_4) = (35, 0, -100, -75)$.

The g.c. values after assembly are $(59.99, -0.03, -119.64, -90.20)$. The exact values are $(60, 0, -120, -90)$. The convergence is achieved after 19 iterations. The assembled configuration is as shown in figure 6.8.

After the deformation, the equilibrium configuration obtained by the system is (70.85, 2.13, -106.96, -100.00). Thus, the slider has moved a distance of 2.13 cm along the link 1 (ref. figure 6.9). The deformable coordinates obtained after deformation are L3 = 4.76 cm and L4

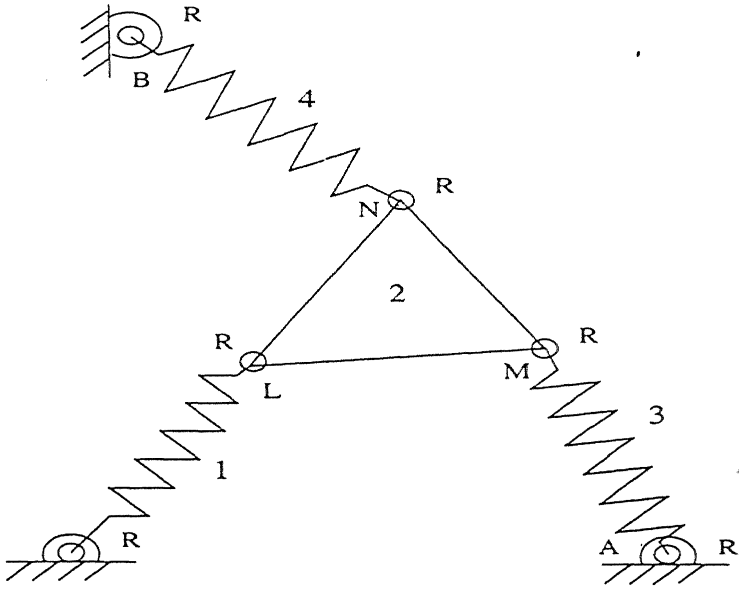


Figure 6.4: Example II

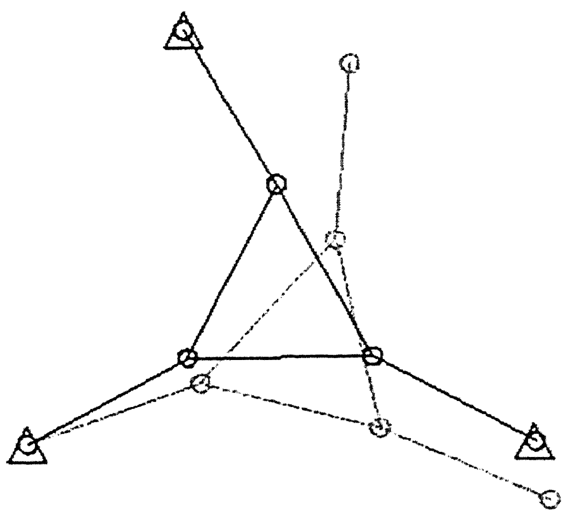


Figure 6.5: Assembly of Example II

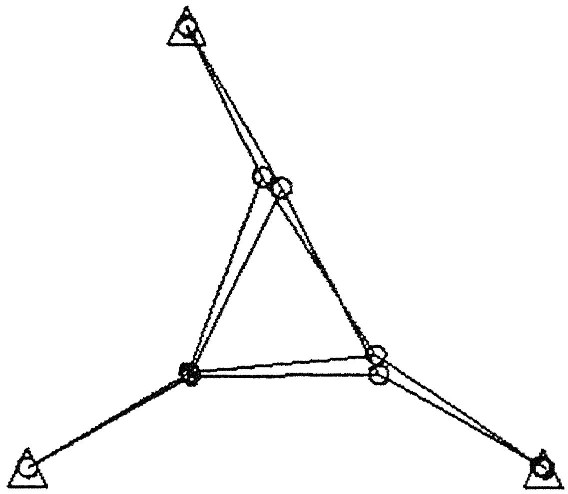


Figure 6.6: Static equilibrium of Example II

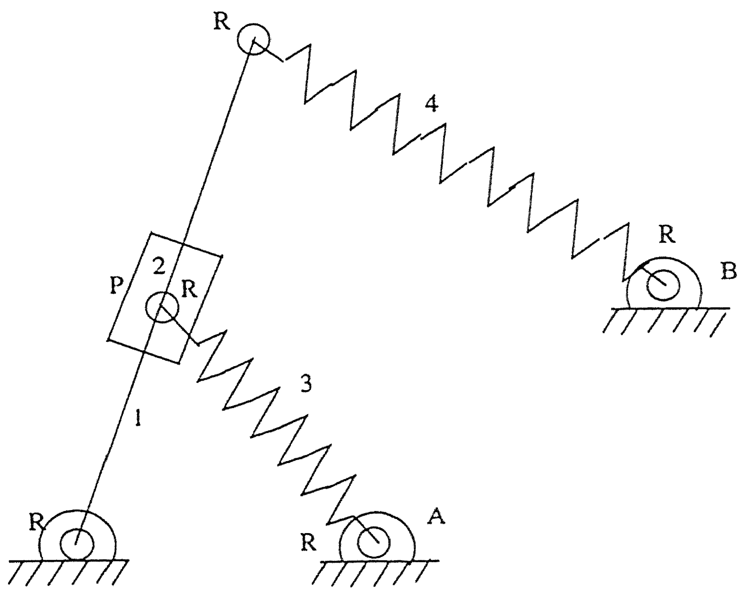


Figure 6.7: Example III

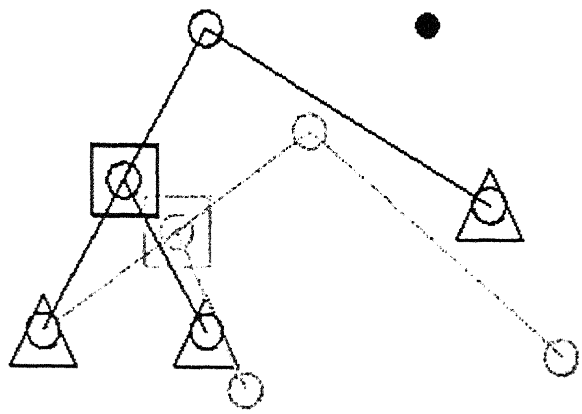


Figure 6.8: Assembly of Example III

= 11.92 cm.

Example IV:This example shows a structure with two sliders and two deformable links (figure 6.10). The parameters of the system are:
Link lengths: L1=L4=10 cm. L3=L6=5 cm.
Spring stiffness: K3=K6=100N/cm.
Position of ground connections: A(5,0,0), C(1.46, 12.19, 0). The angle of slider is 0^0 and the y-intercept is 3.66 cm.
Force data: Force F1 = 100 N acting on link1 at an angle of -60^0 with x-axis.
Moment data: Moment M1= 50N-cm acting in the counter-clock-wise direction, at the c.g. of link 1.
Weights of links: W1=W2=W3=W4=W5=W6=2 N.
User input of orientations $(\phi_1, \phi_2, \phi_3, \phi_4, \phi_5, \phi_6) = (35, 0, -100, -75, 30, 60)$.

The system assembled in 37 iterations giving the configuration as (59.77, -0.05, -119.07, -89.61, 29.69, 75.74). The exact values of the g.c are $(60, 0, -120, -90, 30, 75)$. The assembled configuration is as shown in the figure 6.11.

Figure 6.12 shows the configuation of the system after equilibrium. The final configuration is $(64.21, -0.36, -118.69, -96.45, 31.96, 68.09)$. The lengths of deformable members obtained are L1 = 5.09 cm amd L2 = 4.22 cm. The algorithm converged in 3 iterations.

Example V: This system (figure 6.13) has a torsion spring at the base of link 1. It has also a slider at the end of loop. The parameters of the system are:
Link lengths: L1=L4=10 cm. L3=5 cm.
Spring stiffness: torsion spring stiffness = 2000 N-cm/rad.
Position of ground connections: A(6.82,0.84,0). The angle of slider with x-axis is zero degrees and the y-intercept is 8.83 cm.
Force data: Force F1=100 N acting on link 5 at an angle of 180^0 with x-axis.

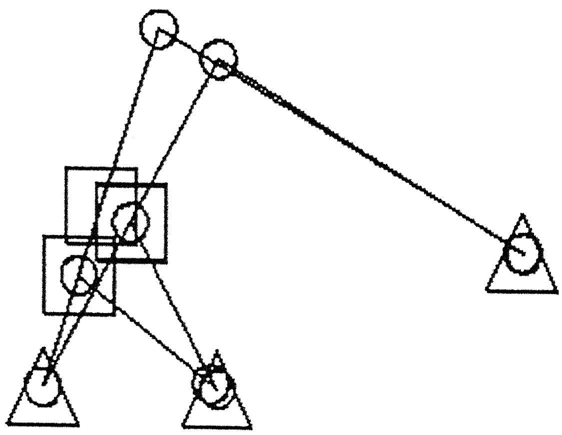


Figure 6.9: Static equilibrium of Example III

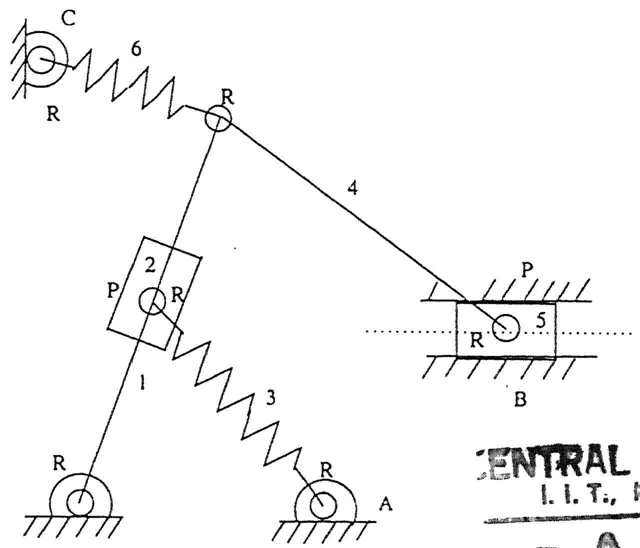


Figure 6.10: Example IV

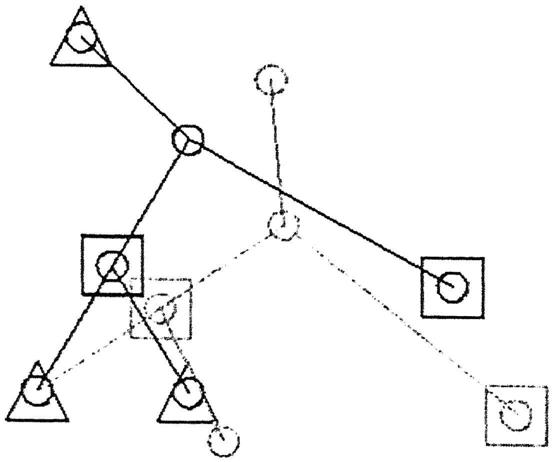


Figure 6.11: Assembly of Example IV

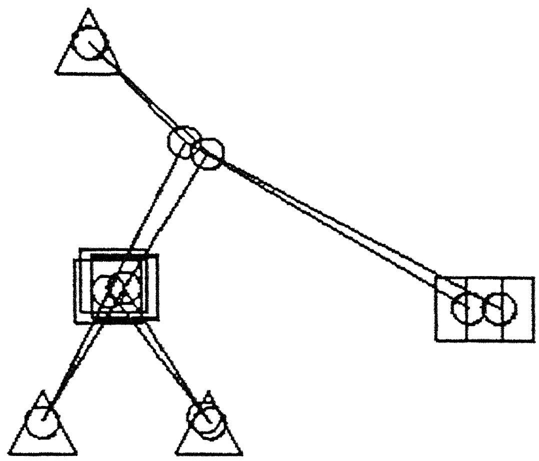


Figure 6.12: Static equilibrium of Example IV

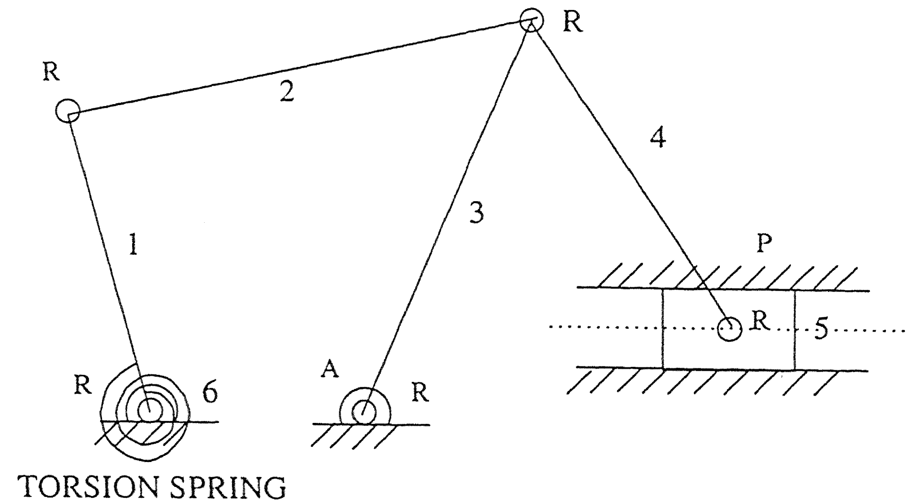


Figure 6.13: Example V

User input of orientations $(\phi_1, \phi_2, \phi_3, \phi_4, \phi_5) = (110, -105, -135, -45, 30)$.

The algorithm converges in 56 iterations to give the assembled position as (116.82, -103.54, -135.98, -41.24, 29.92). The exact values are $(115, -105, -135, -40, 30)$. Refer figure 6.14.

The equilibrium position is $(150.72, -117.86, -127.68, -77.07, 33.39)$ and the deformable coordinate i.e. the twist of the torsion spring is 150.72 degrees, which is same as the orientation of link 1. The deformed positon is as shown in figure 6.15.

Example VI: figure 6.16 shows a five bar parallelogram mechanical system. It has three deformable members. The parameters of the system are:
Link lengths: L1=L3=L4=10 cm. L2=20 cm.
Spring stiffness: K1=K3=K4= 2000 N-cm/rad.
Position of ground connections: A(10,0,0),B(10,0,0).
Force data: Force F1= 50 N acting on link 1 along the positive x-axis.
User input of orientations $(\phi_1, \phi_2, \phi_3, \phi_4) = (40, -30, -110, -110)$.

The assembled configuration obtained is $(50.92, -51.00, -128.92, -128.80)$. It is as shown in figure 6.17. This is basically a singular configuration as discussed in section 2.8. Hence while performing the equilibrium analysis, the program exits and displays the message: *EXITING AS THE CONFIGURATION IS SINGULAR.*

Spatial mechanical systems:

Example VII: This system (figure 6.18) consists of four spherical joints, one revolute and one screw joints. There are two deformable members, link 2 and link 4. The parameters of the system are:
Link lengths: L2=L3=L4=10 cm.
Spring stiffness: K2=200 N/cm, K4=150 N/cm.
Position of ground connections: A(12,0,7), B(13,8,4).
Force data: Force F1=400 N acting on link 2 at an angle along the negative x-axis.
Weights of links: W1=W2=W3=W4=2N.

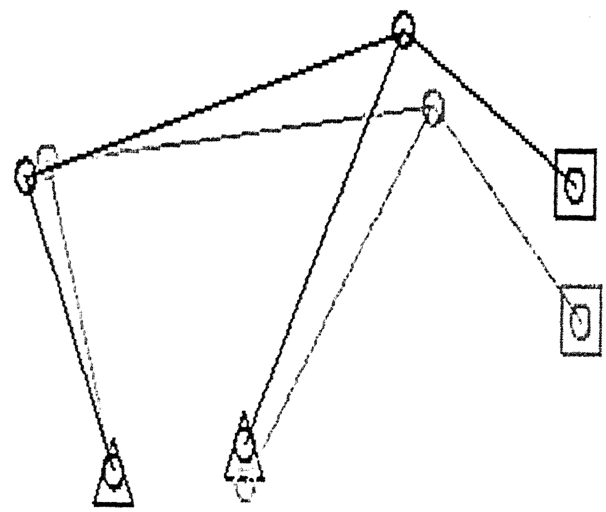


Figure 6.14: Assembly of Example V

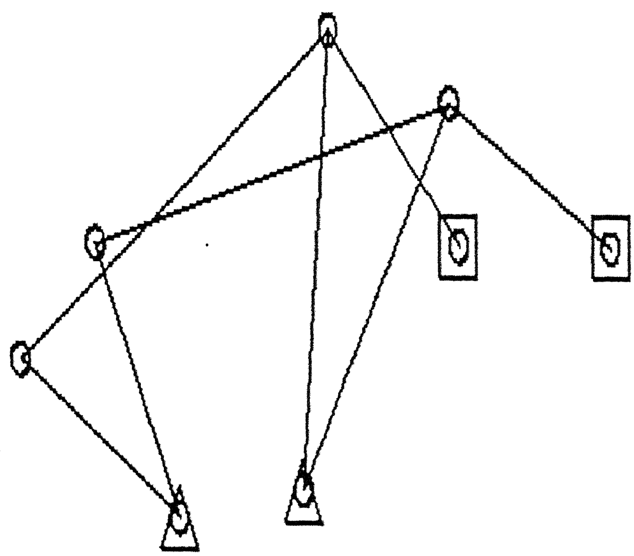


Figure 6.15: Static equilibrium of Example V

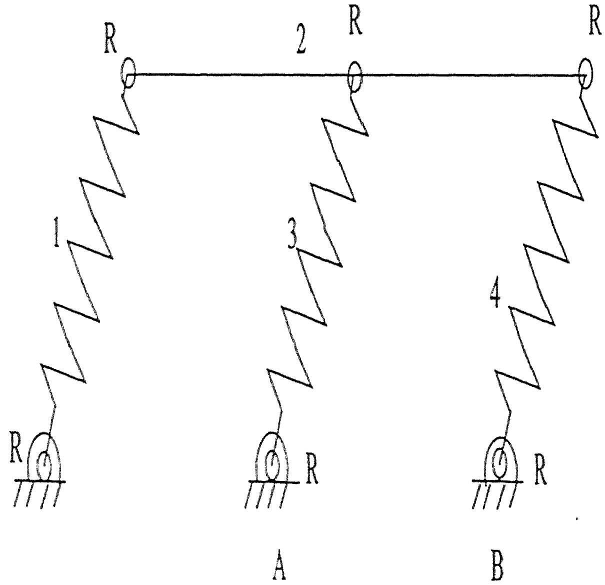


Figure 6.16: Example VI

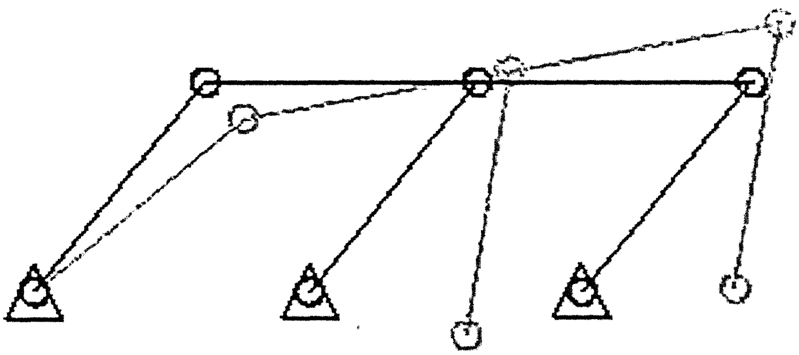


Figure 6.17: Assembly of Example VI

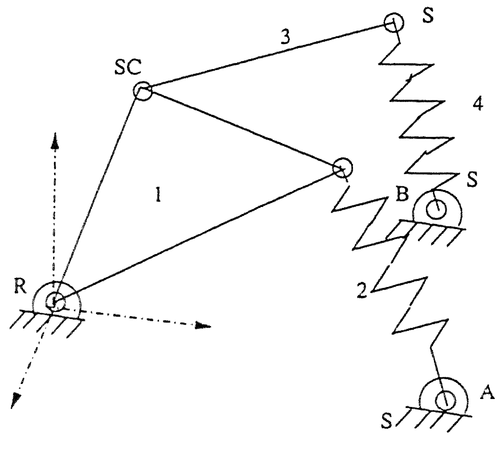


Figure 6.18: Example VII

The link 1 is connected to the ground by a revolute joint. Hence it has 1 g.c i.e. ϕ_1. Links 2 and 4 have two spherical joints each and hence have two g.c. ϕ_2, ϕ_3 and ϕ_5, ϕ_6 respectively, as they each have a redundant DOF. Link 3 has a screw joint connecting the previous link along the loop. Hence, it has 1 g.c. ϕ_4.
User input of orientations $(\phi_1, \phi_2, \phi_3, \phi_4, \phi_5, \phi_6) = (-65, 20, 50, 20, -20, -20)$. The lead of the screw is 0.1 cm/revolution.

The assembled configuration in this case shown in figure 6.19 is (33.74, 89.24, 134.01, -13.87, -104.77, 21.00). The convergence is obtained after 32 iterations.

The equilibrium configuration of the deformed system is (41.22, 83.72, 138.2, -23.94, -92.39, 23.23). The deformable coordinates are L2 = 10.89 cm, L4 = 9.98 cm. Number of iterations = 3. The system is shown in figure 6.20.

Example VIII: Figure 6.21 shows a system with two deformable members link 3 and link 4. It also has a slotted link 1 on which a slider is present. The parameters of the system are:
Link lengths: L1=L4=10 cm. L3=5 cm.
Spring stiffness: K3=K4=200 N/cm.
Position of ground connections: A(5,0,3), B(10,0,-6).
Force data: Force F1=600 N acting on link 1 along the y-axis.
Moment data: M_z= 1500N-cm acting counter-clock-wise at the c.g. of link 1.
Weights of links: W1=W2=W3=W4=2N.
User input of orientations $(\phi_1, \phi_2, \phi_3, \phi_4, \phi_5, \phi_6) = (60, 0, -120, 10, -120, -10)$.

The assembled configuration is as shown in the figure 6.22. The convergence is obtained in 16 iterations with the correct initial configuration as (48.66, -0.003, -115.61, 34.92, -115.36, -33.09).

The equilibrium configuration is achieved by the system after 3 iterations (figure 6.23). The final configuration is $(59.11, -0.43, -115.13, 32.34, -119.53, -31.33)$. The deformable coordinates are L3 = 4.96 cm and L4 = 11.79 cm.

Example IX: Figure 6.24 shows a spatial linkage with one deformable link, link 3. It also has a link 2 of tetrahedral shape. The parameters of the system are:

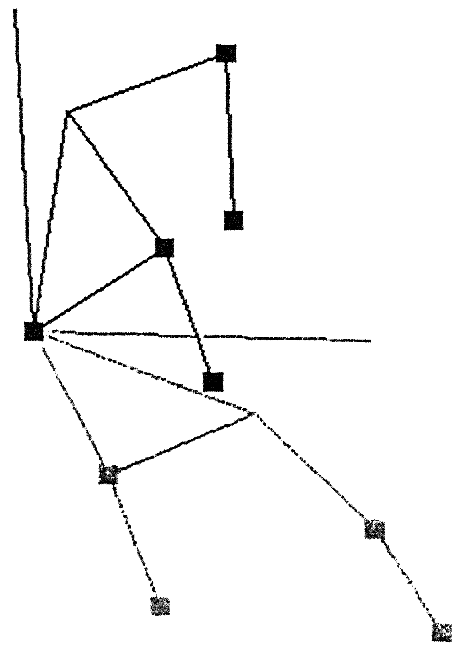


Figure 6.19: Assembly of Example VII

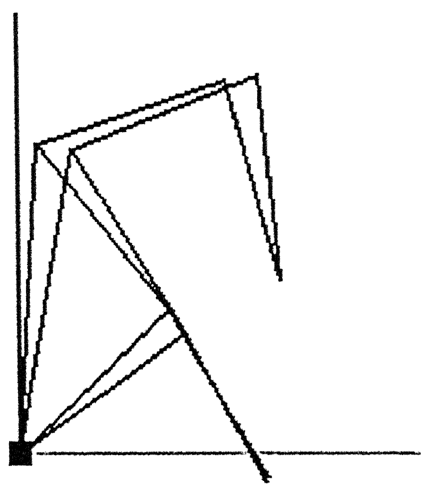


Figure 6.20: Static Equilibrium of Example VII

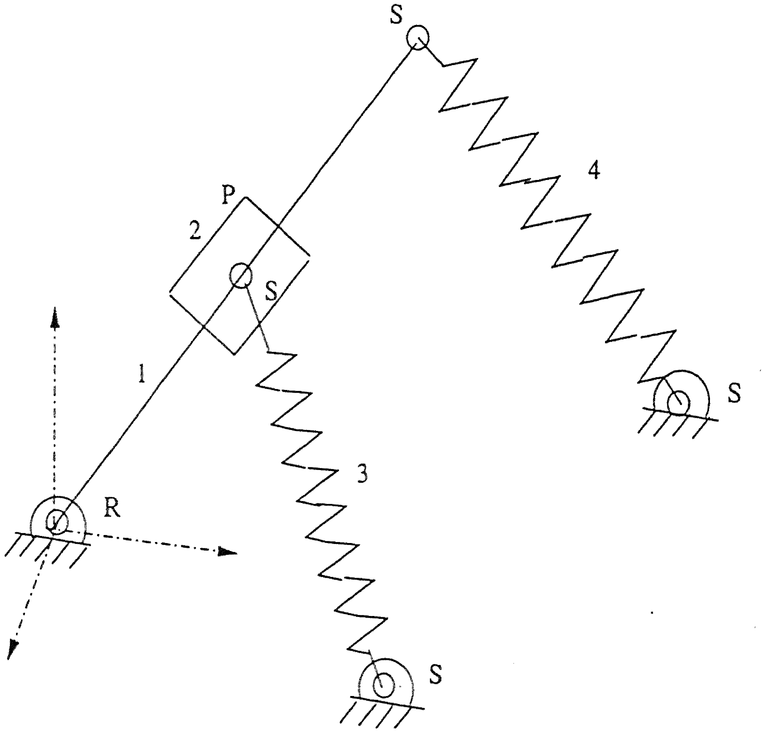


Figure 6.21: Example VIII

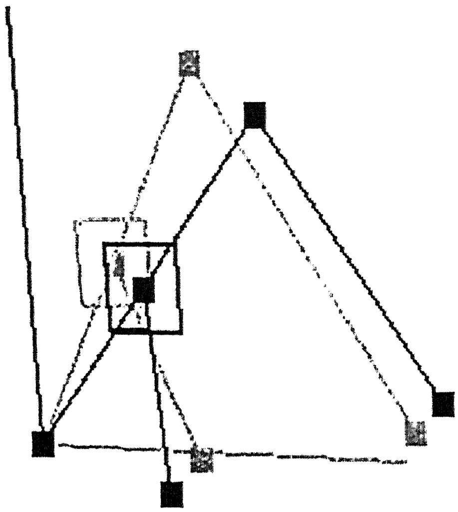


Figure 6.22: Assembly of Example VIII

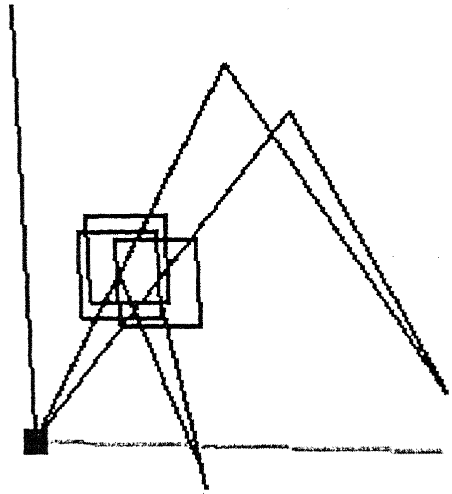


Figure 6.23: Static equilibrium of Example VIII

Link lengths: L1=L3=L4=L5=10 cm.
Spring stiffness: K3=200 N/cm.
Position of ground connections: A(21,0,2), B(12,0,-8), C(8,23,0). The direction cosines of the revloute joint C are (0.3,0.5,0.81).
Force data: Force F1=400 N acting on link 2 along the x-axis.
Weights of links: W1=W2=W3=W4=W5=2N.
User input of orientations $(\phi_1, \phi_2, \phi_3, \phi_4, \phi_5, \phi_6, \phi_7, \phi_8, \phi_9, \phi_{10}, \phi_{11})$ = (60, -60, -20, 0 -60, 10, -50, 10, -10, 45, 10).

The algorithm of assembly converges in 38 iterations to give the correct configuration as (33.78, -38.07, -32.45, -78.96, -88.81, -17.29, -65.54, -40.18, 8.48, 133.77, 16.80). This configuration is as shown in figure 6.25.

After deformation the configuration becomes (33.95, -42.45, -15.07, -68.13, -73.13,-11.95, -73.59, -49.49, -12.69, 144.31, -1.41). The deformable member is L3 = 6.43 cm (figure 6.26). The algorithm converges in 6 iterations.

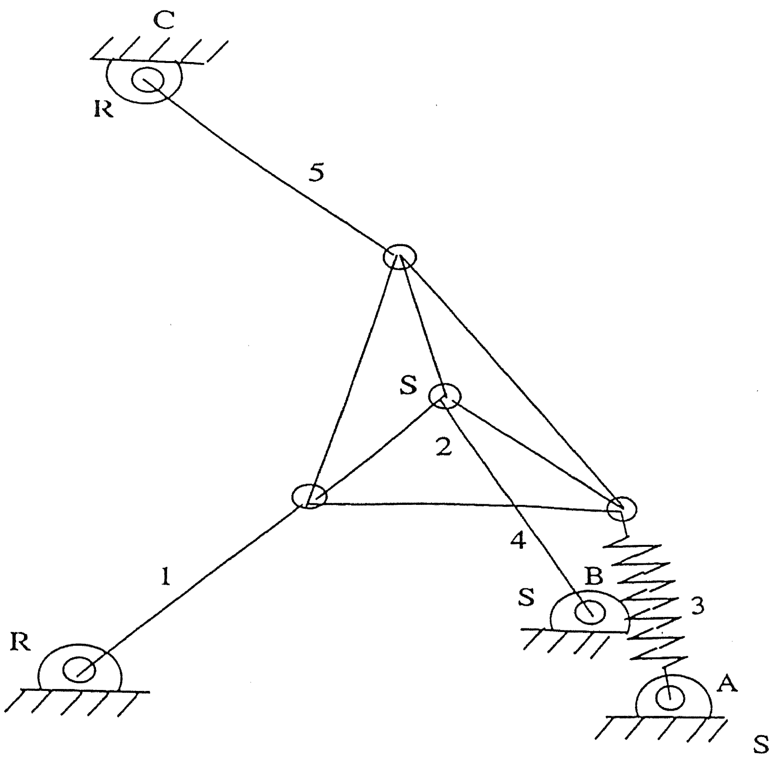


Figure 6.24: Example IX

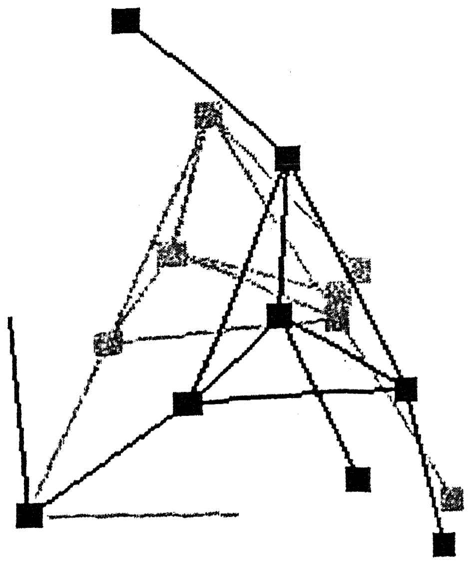


Figure 6.25: Assembly of Example IX

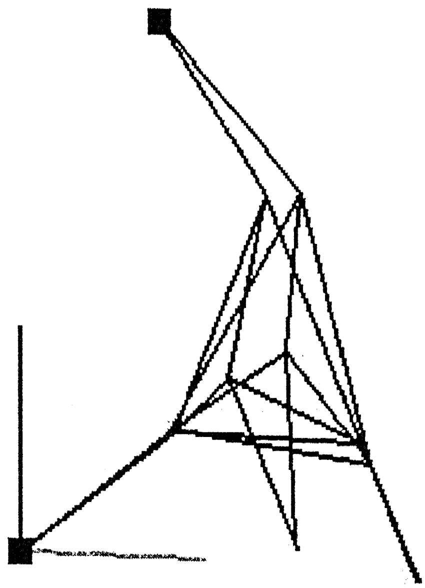


Figure 6.26: Static equilibrium of Example IX

Chapter 7

Conclusion

7.1 Summary

In this thesis, the static equilibrium analysis of compliant structures is performed. Both planar and spatial statically determinate structures are considered. The systems considered are user defined and not of a particular category.

First of all, closed loops are found out in the system. The kinematic constraints are developed in terms of the generalized coordinates. These constraint equations depend upon the joints at the end of every loop. The generalized coordinates used are the relative orientations of every link with respect to the previous link in the loop. In case of complicated planar and spatial systems, providing accurate values of these relative orientations is a difficult task for the user. Hence, the user is expected to enter only an approximate estimate of the g.c.

For generating both the rigid and deformable constraint equations, loop closure equations are used. The assembly of the system is done using the optimization technique to get the correct initial configuration of the system.

Deformable members are modeled as linear and torsion springs. The deformable constraint equations are found out in terms of the generalized coordinates. The external loading is converted into the generalized load. Then, the force equilibrium equation is used to get the equilibrium configuration of the system.

A check is also provided to detect the singular cases. These are the cases in which there is an unconstrained mobility of the system.

Some results are generated for both planar and spatial cases, with various kinematic joints and the results are found to be satisfactory. The convergence observed is fast.

Thus, from the results obtained, we can conclude that our method used for static equilibrium analysis is computationally fast, efficient and accurate. The main reason for this is the use of a lesser number of generalized coordinates for the system and also the use of loop-closure equations which are computationally simple.

7.2 Future Scope

The systems we have dealt with are statically determinate. Hence the next step in this research will be to consider the statically indeterminate cases also. In this case, the force equilibrium approach in itself will not be sufficient and we will have to go for other approaches like the minimum total energy approach.

Also, the systems we have tackled are more or less idealized, i.e. we have neglected the friction effects, damping and the joint clearances. Also we have modeled the deformable members as linear elastic elements. But the real life mechanical systems may have deformable members which are elasto-plastic and also with nonlinear deformation rates. Hence modeling the practical deformable members will also be a important task in the future research.

In our approach, if we detect any singularity in the configuration, then the further analysis is stopped. Hence, predicting the way of deformation of such singular configurations will be another area of research.

Bibliography

[1] B. A. Salamon, Mechanical advantage aspects in complaint mechanisms design, M.S. Thesis,Purdue University, 1989.

[2] L. L. Howell and A. Midha," A loop-closure theory for the analysis and synthesis of compliant mechanisms", Journal of Mechanical Design, Vol.118, pp. 121-125, March 1996.

[3] A. Midha, L. L. Howell, T. W. Norton,"Limit positions of compliant mechanisms using the pseudo-rigid-body model concept", Mechanism and machine theory, Vol 35, pp. 99-115, 1999.

[4] M. Hac,"Dynamics of flexible mechanisms with mutual dependence between rigid body motion and longitudinal deformation of links", Mechanism and machine theory, Vol. 30, No. 6, pp. 837-847, 1995.

[5] G. G. Lowen and C. Chassapis,"The elastic behavior of linkages: an update", Mechanism and machine theory, Vol. 21, No. 1, pp. 33-42, 1996.

[6] B. Dasgupta, M.M. Deshpande and T.S. Mruthyunjaya,"A static equilibrium formulation for mechanical systems with compliant members", Tenth world congress on the theory of machines and mechanisms, Oulu, Finland, June 20-24, 1999.

[7] G. N. Sandor and Arthur G. Erdman, Advanced mechanism design: Analysis and synthesi-Vol. 1, Prentice Hall Inc., 1984.

[8] E. J. Haug, Computer aided kinematics and dynamics of mechanical systems- Vol. 1: Basic methods, Allyn and Bacon, 1989.

[9] S. S. Sastry, Introductory methods of numerical analysis, Prentice Hall of India pvt. Ltd, 1998.

[10] K. Deb, Optimization for engineering design: algorithms and examples, Prentice Hall of India Pvt. Ltd, 1996.

[11] J. J. Craig, Introduction to robotics: mechanics and control, Addison-Wesley Publishing Company, 1999.

[12] A. Ghosh, A. K. Mallik, Theory of mechanisms and machines, Affiliated East-West Press Pvt. Ltd, 1976.

[13] E. J. Haug, Intermediate dynamics, Prentice Hall, 1992.

[14] W. H. Press, S. A. Teukolsky, B. P. Flannery and W. T. Vetterling, Numerical Recipes in C, Cambridge University Press, 1989.

Appendix A

The user input for a sample problem is given below. The sample problem is as shown in figure 7.1. The information asked by the software is given in italics. Input provided by the user is given in bold type. Explanation is given in the normal type.

Enter the total no. of links (including base) & no. of Rigid links

5 3

Rigid links are to provided excluding the ground.

what is domain ? enter 1– planar ,2– spatial **1**

Enter the flexible link no. & corresponding Stiffness. Is it a torsion spring ?

ENTER 1 - YES , 0 - NO **0**

flexible link no.= **4**

k4= **100**

The units of stiffness are N/cm.

Enter the total number of joints in the system **6**

enter the connectivity of all joints (including base) with the smaller link no. 1st & larger 2nd (for ground connection, zero should always be 2nd, except for the joint at global reference frame).

Also enter number for type of joint

0 1 1
1 2 1
2 3 1
3 0 2
1 4 1
4 0 1

Zero is entered first for the joint at the global reference frame for indicating that the joint is used as the global origin. The numbers for various types of joints are:

revolute-1, prismatic-2, spherical-3, cylindrical-4, universal-5, screw-6.

O.K. Statically Determinate. Go ahead.

The static determinacy of the system is checked by the code. If the system is statically indeterminate, the user is informed the same and the further process is stopped.

for link 1, enter the no. of joints by it is connected to other links **3**

enter the links connected through these connections **0 2 4**

enter the base of link **0 2**

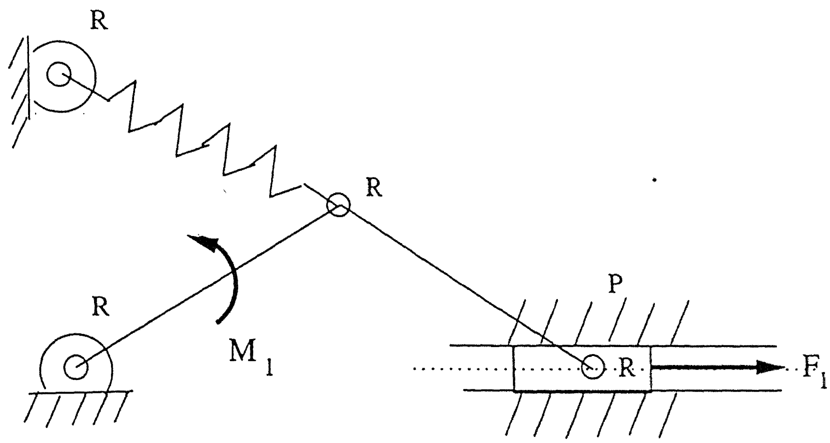


Figure 7.1: Sample compliant mechanical system

Base of the link is given by any two joints of the corresponding link. It acts as a reference that the g.c. values (all orientations of the link), provided by the user and obtained after the equilibrium analysis, are with respect to this base of the link.

enter the x, y, z distances of joints from cg:
link 0 *x*=-5 *y*=0 *z*=0
link 2 *x*=5 *y*=0 *z*=0
link 4 *x*=5 *y*=0 *z*=0
All the distances are entered in cm.
for link 2, enter the no. of joints with it is connected to other links 3
enter the links connected through these connections 1 4 3
enter the base of link 1 3
enter the x, y, z distances of joints from cg:
link 1 *x*=-5 *y*=0 *z*=0
link 4 *x*=-5 *y*=0 *z*=0
link 3 *x*=5 *y*=0 *z*=0
for link 3, enter the no. of joints with it is connected to other links 2
enter the links connected through these connections 2 0
enter the base of link 2 0
enter the x, y, z distances of joints from cg:
link 2 *x*=-1 *y*=0 *z*=0
link 0 *x*=1 *y*=0 *z*=0
for link 4 enter the no. of joints with it is connected to other links 3
enter the links connected through these connections 1 2 0
enter the base of link 1 0
enter the x, y, z distances of joints from cg:
link1 *x*=-5 *y*=0 *z*=0
link 2 *x*=-5 *y*=0 *z*=0
link 0 *x*=5 *y*=0 *z*=0
enter the absolute angle of slider with base 0

what is it's y-intercept with base reference **0**

For a slider, slider angle with respect to the global reference system and y-intercept of the direction of motion of the slider are to be provided.

Enter the accurate values of x, y, z coordinates of ground connection to 4

0 12 0

In spatial system, if revolute joint is connected to the ground, then along with it's x, y and z coordinates, the direction cosines of its axis of revolution are also needed.

enter the weight of every link:

*link 1=***5** *link 2=***5** *link 3=***5** *link 4=***5**

Weight of link is entered in N.

enter the no. of locations at which the forces are applied **1**

force no 1, at link no=? **4**

magnitude of force= **200**

The force input is in N.

angles of force with respect to global reference system :

angle 1= **0** *angle 2=* **90** *angle 3=* **90**

location w.r.t. the cg of the link: x= **2** *y=* **0** *z=* **0**

enter the total no. of links on which moments are acting **1**

moment no1 at the c.g. of link no=? 1

magnitude of moment= **100**

The moments are to be entered in N-cm.

enter the angle w.r.t. 0 of link 1 **20**

enter the angle w.r.t. 1 of link 2 **-60**

enter the angle w.r.t. 2 of link 3 **30**

enter the angle w.r.t. 1 of link 4 **120**

All the angles are to be entered in degrees.

APPENDIX B

A library of concatenated homogeneous transformation matrices for every kinematic joint is given below.

Prismatic joint:

$$T(\Delta L, 0) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & \Delta L \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

where ΔL is the relative displacement between the links connected by the prismatic joint.

Cylindrical joint:

$$T(\Delta L, \phi) = T(\Delta L, 0) * R(0, \phi)$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & \Delta L \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} cos\phi & -sin\phi & 0 & 0 \\ sin\phi & cos\phi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

where ΔL is the relative displacement between links and ϕ is the relative angle between the two.

Spherical joint:

The spherical joint allows freedom of rotation around all the three axes of one link relative to another. Typically these rotations are described using Euler angles. Here the Z-Y-X Euler angles are used.

Thus, for spherical joint, we have,

$$T(L, \phi_z, \phi_y, \phi_x) = T(L, \phi_z) * R(\phi_y) * R(\phi_x)$$

$$= \begin{bmatrix} cos\phi_z & -sin\phi_z & 0 & L \\ sin\phi_z & cos\phi_z & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} cos\phi_y & 0 & -sin\phi_y & 0 \\ 0 & 1 & 0 & 0 \\ sin\phi_y & 0 & cos\phi_y & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & cos\phi_x & -sin\phi_x & 0 \\ 0 & sin\phi_x & cos\phi_x & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Screw joint:

A screw joint is represented by

$$T(k,\phi) = \begin{bmatrix} cos\phi_z & -sin\phi_z & 0 & 0 \\ sin\phi_z & cos\phi_z & 0 & 0 \\ 0 & 0 & 1 & \Delta L + \frac{k\phi}{2\pi} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Where ϕ is the relative angle between the links and k is lead of the screw.

Where, ϕ is the relative angle between the links and k is lead of the screw.

In this way, every type of joint can be modeled as the function of generalized coordinates (orientations, relative displacements), using the homogeneous transformations matrices.

Appendix C

Functions used in the code:

Following is the list of functions used in the code. For every function, the required input and its output is given.

The C++ code has two classes, mech and Matrix. The mech class includes all the functions required for assembly and static equilibrium analysis. The Matrix class includes various elementary matrix and vector operation functions and functions for numerical techniques dealing with linear equations.

Class mech

Input functions:

getlink() - gets the total number of links, domain of system (planar or spatial)

flexdata() - gets the information of deformable members i.e. link number, stiffness, nature of spring (linear or torsional)

get_connect() - gets the connectivity of all the joints in the system and their type of joint. It checks the system for static determinacy. If system is statically indeterminate, exits the code, stopping the procedure.

link_data() - gets the number of connections of every link and the links connected through these links. Also the base of the link and the position of all joints with respect to the c.g. of the link is input.

base_coord() - gets the x, y, z coordinates of the ground connections, direction of slider movement, axis of revolution of spatial revolute joint.

force_details() - gets the weight of every link, the locations and magnitudes of forces and moments.

origins() - determines the number of loops in the system and gives the connectivity of the system along the loops.

angle_data() - based on the connectivity obtained in *origins()*, gets the relative orientations of the every link with respect to previous link along the loop. These are stored as the generalized coordinates of the system.

Other functions:

global_declaration() - It is used for dynamic memory allocation of all the global variables and objects required in all the next functions.

constraints() -

Input: generalized coordinates, connectivity and type of joints along the loop, link geometry.

Output: constraint equations $f(X)$.

grad_constraint() -

Input: generalized coordinates, connectivity and type of joints along the loop, link geometry.

Output: Jacobian of rigid constraints $\nabla f(X)$.

grad_sq() -

Input: generalized coordinates, connectivity and type of joints along the loop, link geometry.

Output: Hessian of rigid constraints $\nabla^2 f(X)$.

deformable_cd() -

Input: generalized coordinates, connectivity and type of joints along the loop, link geometry, deformable link numbers.

Output: Deformable constraints $g(X)$.

flexible_grad() -

Input: generalized coordinates, connectivity and type of joints along the loop, link geometry, deformable link numbers.

Output: gradient of deformable coordinates $\nabla g(X)$.

flex_grad() -

Input: generalized coordinates, connectivity and type of joints along the loop, link geometry, deformable link numbers.

Output: gradient of deformable coordinates $\nabla^2 g(X)$.

levenberg()-

Input: $f(X), \nabla f(X), \nabla^2 f(X)$.

Output: generalized coordinates of assembled configuration of the system.

combine_f_and_g() -

Input: $f(X), \nabla f(X), \nabla^2 f(X), g(X), \nabla g(X), \nabla^2 g(X)$.

Output: combined constraints after replacing some of the rigid constraints by deformable coordinates. Combined Jacobian and Hessian are also obtained.

F_load() -

Input: g.c., force locations, force magnitudes, moment locations, moment magnitudes, connectivity and type of joints along the loop, link geometry

Output: F_{load}

equilibrium_eqn() -

Input: combined constraints vector, Jacobian and Hessian matrices

Output: deformable members obtained after the equilibrium analysis

graphic_display() -

Input: connectivity and type of joints along every loop, link geometry, g.c.

Output: x, y, z coordinates of every joint of all the links.

main.c++ - main program of entire code.

CADraw.c - Uses the output of *graphic_display()* and displays the output graphically.

Class Matrix:

Important functions:

LUDEC() - LU decomposition for getting inverse of the matrix.

matrix_rank() - gives the rank of the matrix

gauss() - Gaussian elimination for solving liner equations

The LU decomposition and Gaussian elimination codes are taken from press [14].

A 130868

Date Slip

This book is to be returned on the date last stamped.
